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On Solutions to the Linear Boltzmann Equation with General Boundary Conditions and Infinite-Range Forces

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This paper considers the linear space-inhomogeneous Boltzmann equation in a convex, bounded or unbounded body D with general boundary conditions. First, mild L^1-solutions are constructed in the cutoff case using monotone sequences of iterates in an exponential form. Assuming detailed balance relations, mass conservation and uniqueness are proved, together with an H-theorem with formulas for the interior and boundary terms. Local boundedness of higher moments is proved for soft and hard collision potentials, together with global boundedness for hard potentials in the case of a nonheating boundary, including specular reflections. Next, the transport equation with forces of infinite range is considered in an integral form. Existence of weak L^1-solutions are proved by compactness, using the H-theorem from the cutoff case. Finally, an H-theorem is given also for the infinite-range case.

KEY WORDS: Linear Boltzmann equation; transport equation; initial boundary value problem; boundary conditions; entropy function; H-functional; detailed balance relation; mild L^1-solution; higher moments; infinite-range forces.

0. INTRODUCTION

The linear Boltzmann equation is frequently used for mathematical modeling in physics. This paper studies that space-inhomogeneous transport equation for a distribution function $f(\mathbf{x}, \mathbf{v}, t)$ (describing, for instance, a neutron distribution) depending on a space variable $\mathbf{x} = (x_1, x_2, x_3)$ in a nonmultiplying, nonabsorbing (i.e., purely scattering) body D, and depending on a velocity variable $\mathbf{v} = (v_1, v_2, v_3) \in V = \mathbb{R}^3$ and time variable

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$t \in \mathbb{R}_+$. Here we assume $D = \bar{D}$ to be a closed, bounded or unbounded, (strictly) convex domain in $\mathbb{R}^3$ with (piecewise) C^1-boundary $\Gamma = \partial D$. In absence of external forces the transport equation in strong form is

$$\frac{\partial f}{\partial t}(\mathbf{x}, \mathbf{v}, t) + \mathbf{v} \cdot \operatorname{grad}_{\mathbf{x}} f(\mathbf{x}, \mathbf{v}, t) = (Qf)(\mathbf{x}, \mathbf{v}, t) \tag{0.1}$$

$$\mathbf{x} \in D \backslash \Gamma, \qquad \mathbf{v} \in V, \qquad t \in \mathbb{R}_+$$

supplemented with initial data

$$\lim_{t \downarrow 0} f(\mathbf{x}, \mathbf{v}, t) = F_0(\mathbf{x}, \mathbf{v}), \qquad \mathbf{x} \in D, \qquad \mathbf{v} \in V \tag{0.2}$$

and some boundary conditions.

In earlier papers[11,12] I considered periodic boundary conditions (in $\mathbf{x}$). In the present paper the boundary conditions are chosen (cf. ref. 4 p. 107) as

$$|\mathbf{nv}| \, f(\mathbf{x}, \mathbf{v}, t) = \int_{\mathbf{nv}' > 0} R(\mathbf{x}, \mathbf{v}' \to \mathbf{v}) \, f(\mathbf{x}, \mathbf{v}', t) \, |\mathbf{nv}'| \, d\mathbf{v}' \tag{0.3}$$

$$\mathbf{x} \in \Gamma, \qquad \mathbf{nv} < 0, \qquad t \geqslant 0$$

where $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is the unit, outward, normal vector at $\mathbf{x} \in \Gamma = \partial D$, and R is a given nonnegative function. For instance, in the case of specular reflection, then

$$R(\mathbf{x}, \mathbf{v}' \to \mathbf{v}) = \delta(\mathbf{v} - \mathbf{v}' + 2\mathbf{n}(\mathbf{nv}')) \tag{0.4}$$

where δ is the usual Dirac measure, and in the case of diffuse reflection

$$R(\mathbf{x}, \mathbf{v}' \to \mathbf{v}) = |\mathbf{nv}| \, M(\mathbf{x}, \mathbf{v}) \tag{0.5}$$

where $M(\mathbf{x}, \mathbf{v})$ is a local Maxwell distribution function.

For a nonabsorbing boundary the function R in (0.3) is supposed to satisfy (cf. ref. 4)

$$\int_{\mathbf{nv} < 0} R(\mathbf{x}, \mathbf{v}' \to \mathbf{v}) \, d\mathbf{v} = 1, \qquad \mathbf{x} \in \Gamma, \quad \mathbf{nv}' > 0 \tag{0.6}$$

Sometimes we write the boundary conditions (0.3)

$$f(\mathbf{x}, \mathbf{v}, t) = \int_{\mathbf{nv}' > 0} W(\mathbf{x}, \mathbf{v}' \to \mathbf{v}) \, f(\mathbf{x}, \mathbf{v}', t) \, d\mathbf{v}' \tag{0.7}$$

$$\mathbf{x} \in \Gamma, \qquad \mathbf{nv} < 0, \qquad t \geqslant 0$$

with

$$W(\mathbf{x}, \mathbf{v}' \to \mathbf{v}) = \frac{|\mathbf{nv}'|}{|\mathbf{nv}|} R(\mathbf{x}, \mathbf{v}' \to \mathbf{v}), \qquad \mathbf{nv}' > 0, \quad \mathbf{nv} < 0 \tag{0.8}$$

The collision term in (0.1) can be written (cf. ref. 4)

$$(Qf)(\mathbf{x}, \mathbf{v}, t) = \int_V \int_S [\psi(\mathbf{x}, \mathbf{v}'_*) f(\mathbf{x}, \mathbf{v}', t) - \psi(\mathbf{x}, \mathbf{v}_*) f(\mathbf{x}, \mathbf{v}, t)] \times B(\theta, w)\, d\theta\, d\zeta\, d\mathbf{v}_* \tag{0.9}$$

where $\psi \geqslant 0$ is a known distribution function. Here, $\mathbf{v}$ and $\mathbf{v}_*$ are the velocities before, and $\mathbf{v}'$ and $\mathbf{v}'_*$ are the velocities after, a binary collision. S is the impact plane

$$\{(r, \zeta) : 0 \leqslant r < \hat{R},\ 0 \leqslant \zeta < 2\pi\}$$

which also can be parametrized by the usual solid-angle representation

$$\{(\theta, \zeta) : 0 \leqslant \theta < \hat{\theta},\ 0 \leqslant \zeta < 2\pi\}$$

In the cutoff case, S is bounded, that is, $\hat{R} < \infty$, or $\hat{\theta} < \pi/2$; but in the case of infinite-range forces, S is the whole plane, i.e., $\hat{\theta} = \pi/2$. The function B is given by

$$B(\theta, w) = wr \left| \frac{\partial r}{\partial \theta} \right|$$

where $r = r(\theta, w)$ is computed through the relevant law of interaction, and $w = |\mathbf{v} - \mathbf{v}_*|$. (For details, see refs. 4 and 15; see also ref. 11).

In many cases of physical interest the function $B(\theta, w)$ has a nonintegrable singularity for $\theta = \pi/2$; for instance, with inverse kth power forces, where

$$B(\theta, w) = w^{\gamma} b(\theta) \tag{0.10}$$

with $\gamma = (k-5)/(k-1)$, $3 < k < \infty$, and

$$b(\theta) = O((\pi/2 - \theta)^{-(k+1)/(k-1)}), \qquad \theta \to \pi/2^-$$

(cf. ref. 4 or ref. 15). For that reason most authors have only dealt with the cutoff case, $\hat{R} < \infty$, or $\hat{\theta} < \pi/2$, including forces of finite range in the collision term (for a discussion of such work, see ref. 11).

The purpose of this paper is to prove the existence of solutions to the linear Boltzmann equation with quite general boundary conditions, first in the cutoff case, and then without cutoff using an H-theorem. That generalizes results in refs. 11–13 for the periodic boundary case.

The space domain D is in Sections 1–4 supposed to be bounded and (strictly) convex, but in Section 5, D is allowed to be unbounded. In Section 1, first mild L^1-solutions are constructed in the cutoff case using iterates in an exponential form. Then mass conservation and uniqueness are proved, assuming detailed balance relations inside D and at the boundary. Section 2 contains an H-theorem for the solution from Section 1 in the cutoff case under detailed balance. In Section 3 first local boundedness in time for higher moments is proved for soft and hard collision potentials in the cutoff case, assuming a "nonheating boundary," and then global boundedness of higher moments is proved for hard potentials in the detailed balance case. Section 4 proves the existence of L^1-solutions to the equation in an integral form for the noncutoff case, including infinite-range forces, and also an H-theorem for this case. Finally, in Section 5 the results are generalized to unbounded domains D, giving results about existence, uniqueness, entropy, and higher moments, first in the cutoff case and then also for infinite-range forces.

1. L^1-SOLUTIONS IN THE CUTOFF CASE

In this section the space domain is supposed to be compact (cf. Sections 0 and 5).

In the case of cutoff in the impact parameters, i.e., $\hat{R} < \infty$ or $\hat{\theta} < \pi/2$, the collision term (0.9) in Eq. (0.1) can be separated into two terms, "a gain term" and "a loss term." A common way to write the collision term is[4,11]

$$(Qf)(\mathbf{x}, \mathbf{v}, t) = \int_V K(\mathbf{x}, \mathbf{v}' \to \mathbf{v})\, f(\mathbf{x}, \mathbf{v}', t)\, d\mathbf{v}' - L(\mathbf{x}, \mathbf{v})\, f(\mathbf{x}, \mathbf{v}, t) \quad (1.1)$$

where

$$L(\mathbf{x}, \mathbf{v}) = \int_V K(\mathbf{x}, \mathbf{v} \to \mathbf{v}')\, d\mathbf{v}' \quad (1.2)$$

The collision frequency L is coupled to the functions ψ and B in (0.9) by the relation

$$L(\mathbf{x}, \mathbf{v}) = \int_V \int_S \psi(\mathbf{x}, \mathbf{v}_*)\, B(\theta, w)\, d\theta\, d\zeta\, d\mathbf{v}_* \quad (1.3)$$

where $w = |\mathbf{v} - \mathbf{v}_*|$. [In earlier papers,[11,12] I used another notation with cross sections Σ_s and Σ, where $K(\mathbf{x}, \mathbf{v}' \to \mathbf{v}) = |\mathbf{v}'|\, \Sigma_s(\mathbf{x}, \mathbf{v}' \to \mathbf{v})$ and

$L(\mathbf{x}, \mathbf{v}) = |\mathbf{v}|\ \Sigma(\mathbf{x}, \mathbf{v})$.] We assume (for simplicity) that the collision kernel K vanishes on Γ and outside D, i.e.,

$$K(\mathbf{x}, \mathbf{v}' \to \mathbf{v}) \equiv 0 \tag{1.4}$$

for $\mathbf{x} \in \mathbb{R}^3 \backslash D^0$, where $D^0 = D \backslash \Gamma$. Then also

$$L(\mathbf{x}, \mathbf{v}) \equiv 0, \qquad \mathbf{x} \in \mathbb{R}^3 \backslash D^0, \qquad \mathbf{v} \in V \tag{1.5}$$

Furthermore, assume that

$$F_0(\mathbf{x}, \mathbf{v}) \equiv 0, \qquad \mathbf{x} \in \mathbb{R}^3 \backslash \bar{D}, \quad \mathbf{v} \in V \tag{1.6}$$

Notation. Let

$$\Gamma_+ = \Gamma_+(\mathbf{v}) = \{\mathbf{x} \in \Gamma; \mathbf{n} \cdot \mathbf{v} > 0\}$$

$$\Gamma_- = \Gamma_-(\mathbf{v}) = \{\mathbf{x} \in \Gamma; \mathbf{n} \cdot \mathbf{v} < 0\}$$

where $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is the unit, outward normal.

Let, for given $\mathbf{x} \in D \backslash \Gamma_-$, $\mathbf{v} \in V$,

$$t_b = t_b(\mathbf{x}, \mathbf{v}) = \inf\{s > 0; \mathbf{x} - s\mathbf{v} \in \mathbb{R}^3 \backslash D\} \tag{1.7}$$

representing the time for a particle going from the boundary to the point $\mathbf{x}$ with velocity $\mathbf{v}$.

In this section the linear Boltzmann equation (0.1)–(0.3) with (0.6), (1.1), and (1.2) is studied in two integrated forms, the mild form [Eq. (1.9) below], and the exponential form (1.10), which both formally can be derived from the equations above. Using, for $\mathbf{x} \in D$, $\mathbf{v} \in V$, $t \in \mathbb{R}_+$, the notation

$$\tilde{f}(\mathbf{x}, \mathbf{v}, t) = \begin{cases} F_0(\mathbf{x} - t\mathbf{v}, \mathbf{v}), & 0 \leqslant t \leqslant t_b \\ f(\mathbf{x} - t_b\mathbf{v}, \mathbf{v}, t - t_b), & t > t_b \end{cases} \tag{1.8}$$

where $\mathbf{x}_b \equiv \mathbf{x} - t_b\mathbf{v} \in \Gamma_-(\mathbf{v})$, we have for the *mild* form

$$f(\mathbf{x}, \mathbf{v}, t) = \tilde{f}(\mathbf{x}, \mathbf{v}, t) + \int_0^t (Qf)(\mathbf{x} - (t-s)\,\mathbf{v}, \mathbf{v}, s)\, ds \tag{1.9}$$

and for the *exponential* form

$$\begin{aligned} f(\mathbf{x}, \mathbf{v}, t) = {} & \tilde{f}(\mathbf{x}, \mathbf{v}, t) \exp\left[-\int_0^t L(\mathbf{x} - (t-s)\,\mathbf{v}, \mathbf{v})\, ds\right] \\ & + \int_0^t \exp\left[-\int_\tau^t L(\mathbf{x} - (t-s)\,\mathbf{v}, \mathbf{v})\, ds\right] \\ & \times \int_V K(\mathbf{x} - (t-\tau)\,\mathbf{v}, \mathbf{v}' \to \mathbf{v})\, f(\mathbf{x} - (t-\tau)\,\mathbf{v}, \mathbf{v}', \tau)\, d\mathbf{v}'\, d\tau \end{aligned}$$

$$\mathbf{x} \in D, \qquad \mathbf{v} \in V, \qquad t \in \mathbb{R}_+ \tag{1.10}$$

In connection with the equations above, we also employ the following related problem, with given functions g, F_0, and h:

$$\begin{aligned}
&\frac{\partial f}{\partial t}(\mathbf{x}, \mathbf{v}, t) + \mathbf{v} \cdot \operatorname{grad}_{\mathbf{x}} f(\mathbf{x}, \mathbf{v}, t) + L(\mathbf{x}, \mathbf{v}) f(\mathbf{x}, \mathbf{v}, t) \\
&\qquad = g(\mathbf{x}, \mathbf{v}, t), \qquad \mathbf{x} \in D \backslash \Gamma, \quad \mathbf{v} \in \mathbf{V}, \quad t \in \mathbb{R}_+ \\
&f(\mathbf{x}, \mathbf{v}, 0) = F_0(\mathbf{x}, \mathbf{v}), \qquad \mathbf{x} \in D, \quad \mathbf{v} \in V \\
&f(\mathbf{x}, \mathbf{v}, t) = h(\mathbf{x}, \mathbf{v}, t), \qquad \mathbf{x} \in \Gamma_-(\mathbf{v}), \quad \mathbf{v} \in \mathbf{V}, \quad t \in \mathbb{R}_+
\end{aligned} \tag{1.11}$$

Together with this problem, we also have the two forms analogous to (1.9) and (1.10). The function f is called a *mild* solution of (1.11) if, for $t \in \mathbb{R}_+$, a.e. $(\mathbf{x} + t\mathbf{v}, \mathbf{v}) \in D \times V$

$$\begin{aligned}
&f(\mathbf{x} + t\mathbf{v}, \mathbf{v}, t) + \int_0^t L(\mathbf{x} + s\mathbf{v}, \mathbf{v}) f(\mathbf{x} + s\mathbf{v}, \mathbf{v}, s)\, ds \\
&\qquad = \tilde{F}(\mathbf{x} + t\mathbf{v}, \mathbf{v}, t) + \int_0^t g(\mathbf{x} + s\mathbf{v}, \mathbf{v}, s)\, ds
\end{aligned} \tag{1.12}$$

where

$$\tilde{F}(\mathbf{x}, \mathbf{v}, t) = \begin{cases} F_0(\mathbf{x} - t\mathbf{v}, \mathbf{v}), & 0 \leqslant t \leqslant t_b \\ h(\mathbf{x} - t_b\mathbf{v}, \mathbf{v}, t - t_b), & t > t_b \end{cases}$$

Analogously, f is said to satisfy the *exponential* form of (1.11) if, for $t \in \mathbb{R}_+$, a.e. $(\mathbf{x} + t\mathbf{v}, \mathbf{v}) \in D \times V$,

$$\begin{aligned}
f(\mathbf{x} + t\mathbf{v}, \mathbf{v}, t) &= \tilde{F}(\mathbf{x} + t\mathbf{v}, \mathbf{v}, t) \exp\left[-\int_0^t L(\mathbf{x} + s\mathbf{v}, \mathbf{v})\, ds\right] \\
&\quad + \int_0^t \exp\left[-\int_\tau^t L(\mathbf{x} + s\mathbf{v}, \mathbf{v})\, ds\right] \cdot g(\mathbf{x} + \tau\mathbf{v}, \mathbf{v}, \tau)\, d\tau
\end{aligned} \tag{1.13}$$

We have the following lemma concerning Eqs. (1.12) and (1.13).

Lemma 1.1. Let $L(t) \equiv L(\mathbf{x} + t\mathbf{v}, \mathbf{v}) \in L^1_{\text{loc}}(\mathbb{R}_+)$ and $g(t) \equiv g(\mathbf{x} + t\mathbf{v}, \mathbf{v}, t) \in L^1_{\text{loc}}(\mathbb{R}_+)$. Then f is a mild solution of (1.11) if and only if the exponential form (1.13) holds.

Proof. For the case $0 \leqslant t \leqslant t_b$, see Lemma 1.2 in ref. 11, where I used a Taylor expansion for $\exp(\cdot)$ to get the proof. For $t > t_b$ use, instead of F_0, a function $\tilde{h}(\mathbf{y}, \mathbf{v}, s)$, constant for $\mathbf{y} = \mathbf{x} + s\mathbf{v}$, $0 \leqslant s \leqslant t - t_b$, such that

$$\tilde{h}(\mathbf{y}_b, \mathbf{v}, t - t_b) = h(\mathbf{y}_b, \mathbf{v}, t - t_b)$$

where $\mathbf{y}_b = \mathbf{x} + (t - t_b)\,\mathbf{v} \in \Gamma_-(\mathbf{v})$, $\mathbf{x} \in \mathbb{R}^3 \backslash D$, $\mathbf{v} \in V$. Then [cf. (1.12)], $\tilde{F}(\mathbf{x} + t\mathbf{v}, \mathbf{v}, t) = \tilde{h}(\mathbf{x}, \mathbf{v}, 0)$ for $t > t_b$. For an extensive discussion about solutions to equations in mild and exponential forms, see ref. 14 (see also refs. 3, 9, 10 and 16 concerning solutions satisfying boundary conditions). ∎

To construct solutions to the linear Boltzmann equation with general boundary conditions, *iterate functions* $f_n = f_n(\mathbf{x}, \mathbf{v}, t)$, $n = 0, 1, 2,\dots$, are defined recursively as follows [cf. (0.8)]:

$$f_0(\mathbf{x}, \mathbf{v}, t) \equiv 0, \qquad \mathbf{x} \in \mathbb{R}^3, \quad \mathbf{v} \in V, \quad t \in \mathbb{R}_+$$

$$f_{n+1}(\mathbf{x}_b, \mathbf{v}, t) = \int_{\mathbf{n}\mathbf{v}' > 0} W(\mathbf{x}_b, \mathbf{v}' \to \mathbf{v})\, f_n(\mathbf{x}_b, \mathbf{v}', t)\, d\mathbf{v}'$$

$$\mathbf{x}_b \in \Gamma_-(\mathbf{v}), \qquad \mathbf{n}\mathbf{v} < 0, \qquad t \in \mathbb{R}_+$$

$$\begin{aligned} f_{n+1}(\mathbf{x}, \mathbf{v}, t) &= \tilde{f}_{n+1}(\mathbf{x}, \mathbf{v}, t) \exp\left[-\int_0^t L(\mathbf{x} - (t-s)\,\mathbf{v}, \mathbf{v})\, ds \right] \\ &\quad + \int_0^t \exp\left[-\int_\tau^t L(\mathbf{x} - (t-s)\,\mathbf{v}, \mathbf{v})\, ds \right] \int_V K(\mathbf{x} - (t-\tau)\,\mathbf{v}, \mathbf{v}' \to \mathbf{v}) \\ &\quad \times f_n(\mathbf{x} - (t-\tau)\,\mathbf{v}, \mathbf{v}', \tau)\, d\mathbf{v}'\, d\tau \\ &\qquad \mathbf{x} \in D \backslash \Gamma_-(\mathbf{v}), \qquad \mathbf{v} \in V, \qquad t > 0 \end{aligned} \tag{1.14}$$

where

$$\tilde{f}_{n+1}(\mathbf{x}, \mathbf{v}, t) = \begin{cases} F_0(\mathbf{x} - t\mathbf{v}, \mathbf{v}), & 0 \leqslant t \leqslant t_b \\ f_{n+1}(\mathbf{x} - t_b\mathbf{v}, \mathbf{v}, t - t_b), & t > t_b \end{cases}$$

Let also, for simplicity,

$$f_n(\mathbf{x}, \mathbf{v}, t) \equiv 0, \qquad \mathbf{x} \in \mathbb{R}^3 \backslash D, \quad \mathbf{v} \in V, \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N} \tag{1.15}$$

Now we first formulate a monotonicity result for the iterates.

Lemma 1.2. If F_0, K, and W are nonnegative functions, then the iterates f_n defined by (1.14)–(1.15) satisfy

$$f_{n+1}(\mathbf{x}, \mathbf{v}, t) \geqslant f_n(\mathbf{x}, \mathbf{v}, t), \qquad n \in \mathbb{N}, \quad \mathbf{x} \in \mathbb{R}^3, \quad \mathbf{v} \in V, \quad t \in \mathbb{R}_+ \tag{1.16}$$

Proof (cf. Lemma 1.1 in ref. 11). By induction and (1.14) one finds that the sequence $\{d_n\}_{n=0}^{\infty}$, where $d_n = f_{n+1} - f_n$, together with $\{\tilde{d}_n\}_{n=0}^{\infty}$, $\tilde{d}_n = \tilde{f}_{n+1} - \tilde{f}_n$, are nonnegative. ∎

Then we can formulate an *existence* theorem about mild solutions to the initial-boundary problem. As usual, $L^1_+(D\times V)$ denotes the almost everywhere nonnegative functions in $L^1(D\times V)$.

Theorem 1.3. Assume that $R(\mathbf{x}, \mathbf{v}'\to\mathbf{v})$, $L(\mathbf{x}, \mathbf{v})$, and $K(\mathbf{x}, \mathbf{v}'\to\mathbf{v})$ are nonnegative, measurable functions, such that (0.6), (1.2), and (1.4) hold, and $L(\mathbf{x}, \mathbf{v})\in L^1_{\mathrm{loc}}(D\times V)$.

If $F_0\in L^1_+(D\times V)$, then there exists a global mild solution (i.e., defined for $t>0$) to the problem (0.1)–(0.3) with (1.1) and (1.6). This solution satisfies

$$\int_D\int_V f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v} \leqslant \int_D\int_V F_0(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v}, \qquad t\in\mathbb{R}_+ \tag{1.17}$$

If

$$L(\mathbf{x}, \mathbf{v})\, f(\mathbf{x}, \mathbf{v}, t)\in L^1_+(D\times V) \tag{1.18}$$

then the trace of the solution f satisfies the boundary condition (0.3) for $t\in\mathbb{R}_+$, a.e. $(\mathbf{x}, \mathbf{v})\in\Gamma\times V$.

Proof. Define iterates f_n, $n=0, 1, 2,\ldots$, by (1.14). If, for a given function f_n, $L(\mathbf{x}, \mathbf{v})\, f_n(\mathbf{x}, \mathbf{v}, t)\in L^1(D\times V\times[0, T])$, $T>0$, then by Lemma 1.1. the following (mild) equation holds, for a.e. $\mathbb{x}, \mathbb{v}\in D\times V$, $t\in\mathbb{R}_+$:

$$f_{n+1}(\mathbf{x}, \mathbf{v}, t) + \int_0^t L(\mathbf{x}-(t-\tau)\,\mathbf{v}, \mathbf{v})\, f_{n+1}(\mathbf{x}-(t-\tau)\,\mathbf{v}, \mathbf{v}, \tau)\, d\tau$$

$$= \tilde{f}_{n+1}(\mathbf{x}, \mathbf{v}, t) + \int_0^t\int_V K(\mathbf{x}-(t-\tau)\,\mathbf{v}, \mathbf{v}'\to\mathbf{v})\, f_n(\mathbf{x}-(t-\tau)\,\mathbf{v}, \mathbf{v}', \tau)\, d\mathbf{v}'\, d\tau \tag{1.19}$$

Changing variables $\mathbf{x}\mapsto\mathbf{x}+t\mathbf{v}$ in (1.19) and differentiating along the characteristics, one finds that, for $t\in[0, T]$, a.e. $(\mathbf{x}, \mathbf{v})\in\mathbb{R}^3\times\mathbb{R}^3$,

$$\frac{d}{dt}\left(f_{n+1}(\mathbf{x}+t\mathbf{v}, \mathbf{v}, t)\right) + L(\mathbf{x}+t\mathbf{v}, \mathbf{v})\, f_{n+1}(\mathbf{x}+t\mathbf{v}, \mathbf{v}, t)$$

$$= \int_V K(\mathbf{x}+t\mathbf{v}, \mathbf{v}'\to\mathbf{v})\, f_n(\mathbf{x}+t\mathbf{v}, \mathbf{v}', t)\, d\mathbf{v}' \tag{1.20}$$

Supposing that $L(\mathbf{x}, \mathbf{v})\, f_n(\mathbf{x}, \mathbf{v}, t)\in L^1_+(D\times V\times[0, T])$ and $|\mathbf{n}\mathbf{v}|\, f_n(\mathbf{x}, \mathbf{v}, t)\in L^1_+(\Gamma_+\times V\times[0, T])$, then by *Green's identity*[3,4,9] and a change of variables,

$$\int_D \int_V f_{n+1}(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v} + \int_0^t \int_V \int_{\Gamma_+} f_{n+1}(\mathbf{x}, \mathbf{v}, \tau)\, |\mathbf{nv}|\, d\sigma\, d\mathbf{v}\, d\tau$$

$$+ \int_0^t \int_D \int_V L(\mathbf{x}, \mathbf{v}) f_{n+1}(\mathbf{x}, \mathbf{v}, \tau)\, d\mathbf{x}\, d\mathbf{v}\, d\tau$$

$$= \int_D \int_V F_0(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v} + \int_0^t \int_V \int_{\Gamma_-} f_{n+1}(\mathbf{x}, \mathbf{v}, \tau)\, |\mathbf{nv}|\, d\sigma\, d\mathbf{v}\, d\tau$$

$$+ \int_0^t \int_D \int_V \int_V K(\mathbf{x}, \mathbf{v}' \to \mathbf{v}) f_n(\mathbf{x}, \mathbf{v}', \tau)\, d\mathbf{v}'\, d\mathbf{x}\, d\mathbf{v}\, d\tau \tag{1.21}$$

where $d\sigma$ represents the surface measure on Γ.

Here the first and third terms on the right-hand side are finite by assumption, and the second term can be transformed, using (0.6), (0.8), and (1.14),

$$\int_0^t \int_V \int_{\Gamma_-} f_{n+1}(\mathbf{x}, \mathbf{v}, \tau)\, |\mathbf{nv}|\, d\sigma\, d\mathbf{v}\, d\tau$$

$$= \int_0^t \int_\Gamma \iint_{\substack{\mathbf{nv}<0 \\ \mathbf{nv}'>0}} R(\mathbf{x}, \mathbf{v}' \to \mathbf{v}) f_n(\mathbf{x}, \mathbf{v}', \tau)\, |\mathbf{nv}'|\, d\mathbf{v}'\, d\sigma\, d\mathbf{v}\, d\tau$$

$$= \int_0^t \int_V \int_{\Gamma_+} f_n(\mathbf{x}, \mathbf{v}', \tau)\, |\mathbf{nv}'|\, d\sigma\, d\mathbf{v}'\, d\tau < \infty \tag{1.22}$$

by assumption. Then all three positive integrals on the left-hand side of (1.21) are finite, which gives the induction step. So (1.21) holds for all $n \geqslant 0$.

Now, using (1.2) and Lemma 1.2,

$$\int_0^t \int_D \int_V \int_V K(\mathbf{x}, \mathbf{v}' \to \mathbf{v}) f_n(\mathbf{x}, \mathbf{v}', \tau)\, d\mathbf{v}'\, d\mathbf{x}\, d\mathbf{v}\, d\tau$$

$$- \int_0^t \int_D \int_V L(\mathbf{x}, \mathbf{v}) f_{n+1}(\mathbf{x}, \mathbf{v}, \tau)\, d\mathbf{x}\, d\mathbf{v}\, d\tau$$

$$= \int_0^t \int_D \int_V L(\mathbf{x}, \mathbf{v})[f_n(\mathbf{x}, \mathbf{v}, \tau) - f_{n+1}(\mathbf{x}, \mathbf{v}, \tau)]\, d\mathbf{x}\, d\mathbf{v}\, d\tau \leqslant 0 \tag{1.23}$$

and also, using (1.21) and (1.22),

$$\int_0^t \int_V \int_{\Gamma_-} f_{n+1}(\mathbf{x}, \mathbf{v}, \tau)\, |\mathbf{nv}|\, d\sigma\, d\mathbf{v}\, d\tau$$

$$- \int_0^t \int_V \int_{\Gamma_+} f_{n+1}(\mathbf{x}, \mathbf{v}, \tau)\, |\mathbf{nv}|\, d\sigma\, d\mathbf{v}\, d\tau$$

$$= \int_0^t \int_V \int_{\Gamma_+} [f_n(\mathbf{x}, \mathbf{v}, \tau) - f_{n+1}(\mathbf{x}, \mathbf{v}, \tau)]\, |\mathbf{nv}|\, d\sigma\, d\mathbf{v}\, d\tau \leqslant 0 \tag{1.24}$$

Then, using (1.21), (1.23), and (1.24),

$$\int_D \int_V f_{n+1}(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v} \leqslant \int_D \int_V F_0(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v}, \qquad n \in \mathbb{N} \tag{1.25}$$

By Levi's theorem on monotone convergence there exists a function

$$f(\mathbf{x}, \mathbf{v}, t) = \lim_{n \to \infty} f_n(\mathbf{x}, \mathbf{v}, t), \qquad \mathbf{x} \in D, \quad \mathbf{v} \in V, \quad t \in \mathbb{R}_+ \tag{1.26}$$

which is a mild solution to the linear Boltzmann equation in the exponential form (1.10), and also satisfies the boundary condition (0.3) for $(\mathbf{x}, \mathbf{v}, t) \in \Gamma \times V \times \mathbb{R}_+$, such that one side of (0.3) exists.

To get the existence of a boundary trace of the solution f we use a trace theorem, Proposition 3.3, Chapter XI, in ref. 9. We can formulate it for our purposes: Suppose f and Lf belong to $L^1_+(D \times V)$. Then f has a unique trace $f^{\pm}$. Furthermore, the Green's identity holds for f, if $f^-(\mathbf{x}, \mathbf{v}, t)\, |\mathbf{nv}| \in L^1(\Gamma \times V \times [0, T])$.

Using this proposition together with (1.17) and (1.18), the existence of a trace follows. Furthermore, let $n \to \infty$ in (1.14b) with (0.8); then, by monotone convergence, the solution f satisfies the boundary condition (0.3) for $t \in \mathbb{R}_+$, a.e. $(\mathbf{x}, \mathbf{v}) \in (\Gamma \times V)$. So Theorem 1.3 follows. ▌

Remark. The iterate function $f_{n+1}(\mathbf{x}, \mathbf{v}, t)$ defined in (1.14) has a natural physical meaning. It represents a distribution of particles which have undergone at most n collisions inside D or at the boundary Γ in the time interval $(0, t)$. The difference $f_{n+1} - f_n$ gives the distribution of particles with exactly n collisions. Then $f = \lim_{n \to \infty} f_n$ represents the distribution of particles with at most denumerably many collisions for $t > 0$.

Assumption. In the rest of this paper we suppose that there is a *detailed balance* relation (or reciprocity relation) for binary collisions inside D between particles with density function f and particles with density function ψ, i.e., we assume that there exists a function $E = E(\mathbf{v}) > 0$ such that (cf. ref. 4, p. 170)

$$K(\mathbf{x}, \mathbf{v} \to \mathbf{v}')\, E(\mathbf{v}) = K(\mathbf{x}, \mathbf{v}' \to \mathbf{v})\, E(\mathbf{v}'), \qquad \mathbf{x} \in D \backslash \Gamma, \quad \mathbf{v}, \mathbf{v}' \in \mathbf{V} \tag{1.27}$$

Using (1.1), (1.2), and (1.27), one finds that the function $E = E(\mathbf{v})$ satisfies

$$(QE)(\mathbf{x}, \mathbf{v}, t) \equiv 0 \tag{1.28}$$

so $E(\mathbf{v})$ is an equilibrium solution to Eq. (0.1) if $F_0(\mathbf{x}, \mathbf{v}) = E(\mathbf{v})$ and if $E(\mathbf{v})$ satisfies the boundary condition (0.3). Another way to formulate the detailed balance relation (1.27) is

$$\psi(\mathbf{x}, \mathbf{v}_*)\, E(\mathbf{v}) = \psi(\mathbf{x}, \mathbf{v}'_*)\, E(\mathbf{v}') \tag{1.29}$$

An important example with detailed balance is given by a local Maxwellian function

$$\psi(\mathbf{x}, \mathbf{v}_*) = X(\mathbf{x}) \exp(-cm_* |\mathbf{v}_*|^2)$$

where X is a function of the space variable $\mathbf{x}$, c is a positive constant, and m_* is the mass for a particle with density function ψ. Then (1.29) holds with

$$E(\mathbf{v}) = a \exp(-cm |\mathbf{v}|^2)$$

where a is a positive constant and m is the mass of a particle. This is so because of the energy conservation law for a binary collision.

In the following we also assume that there exists a function $E^b = E^b(\mathbf{x}, \mathbf{v}) > 0$, giving a *detailed balance* relation at the *boundary*, which can be written[4]

$$|\mathbf{nv}'| \, R(\mathbf{x}, \mathbf{v}' \to \mathbf{v}) \, E^b(\mathbf{x}, \mathbf{v}') = |\mathbf{nv}| \, R(\mathbf{x}, -\mathbf{v} \to -\mathbf{v}') \, E^b(\mathbf{x}, -\mathbf{v})$$
$$\mathbf{nv}' > 0, \qquad \mathbf{nv} < 0 \tag{1.30}$$

One finds, by straightforward calculations using (0.6), that such a function $E^b(\mathbf{x}, \mathbf{v})$ satisfies the boundary condition (0.3). We assume in the following that [cf. (1.27)]

$$E^b(\mathbf{x}, \mathbf{v}) \equiv E(\mathbf{v}), \qquad \mathbf{x} \in \Gamma, \quad \mathbf{v} \in V \tag{1.31}$$

Then $E = E(\mathbf{v})$ is a stationary solution to the linear Boltzmann equation in the strong form (0.1) with (0.2) and (0.3), and also to the equation in the mild form (1.9) and in the exponential form (1.10).

In the case of detailed balance (1.27) and (1.30) with (1.31) we can now prove that equality holds in (1.17), giving mass conservation, and also that the solution in Theorem 1.3 is unique (in the relevant L^1-space).

Theorem 1.4. Assume that the detailed balance relations (1.27) and (1.30) with (1.31) hold, where $E(\mathbf{v}) \in L^1_+(D \times V)$ is an equilibrium solution satisfying

$$K(\mathbf{x}, \mathbf{v} \to \mathbf{v}') \, E(\mathbf{v}) \in L^1_+(D \times V \times V) \qquad \text{and} \qquad |\mathbf{nv}| \, E(\mathbf{v}) \in L^1_+(\Gamma \times V) \tag{1.32}$$

(A) If $f = f(\mathbf{x}, \mathbf{v}, t)$ is the solution given in Theorem 1.3, then

$$\int_D \int_V f(\mathbf{x}, \mathbf{v}, t) \, d\mathbf{x} \, d\mathbf{v} = \int_D \int_V F_0(\mathbf{x}, \mathbf{v}) \, d\mathbf{x} \, d\mathbf{v}, \qquad t \in \mathbb{R}_+ \tag{1.33}$$

(B) Moreover, if $\bar{f} = \bar{f}(\mathbf{x}, \mathbf{v}, t)$ is a (mild) solution to the problem (0.1)–(0.3) satisfying the exponential form (1.10) and

$$\int_D \int_V \bar{f}(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v} \leqslant \int_D \int_V F_0(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v} \tag{1.34}$$

then

$$\bar{f}(\cdot, \cdot, t) = f(\cdot, \cdot, t) \qquad \text{a.e. in } D \times V, \quad t \in \mathbb{R}_+$$

Proof. (A) We use a cutoff in the initial value function,

$$F_0^p(\mathbf{x}, \mathbf{v}) = \min(F_0(\mathbf{x}, \mathbf{v}), p \cdot E(\mathbf{v})), \qquad p = 1, 2, 3, \ldots \tag{1.35}$$

and construct the iterate functions $f_n^p(\mathbf{x}, \mathbf{v}, t)$ for $n = 1, 2, 3, \ldots$, using the initial function F_0^p. By induction and elementary calculations one finds that

$$f_n^p(\mathbf{x}, \mathbf{v}, t) \leqslant p \cdot E(\mathbf{v}), \qquad \mathbf{x} \in D, \quad \mathbf{v} \in V, \quad t \in \mathbb{R}_+, \quad n, p \in \mathbb{N} \tag{1.36}$$

By Theorem 1.3 the increasing sequence $\{f_n^p\}_{n=1}^{\infty}$ has a pointwise limit (when $n \to \infty$) satisfying

$$f^p(\mathbf{x}, \mathbf{v}, t) = \lim_{n \to \infty} f_n^p(\mathbf{x}, \mathbf{v}, t) \leqslant p \cdot E(\mathbf{v}), \qquad p \in \mathbb{N} \tag{1.37}$$

Now we use Eq. (1.21) for the iterates f_n^p,

$$\begin{aligned} &\int_D \int_V f_{n+1}^p\, d\mathbf{x}\, d\mathbf{v} + \int_0^t \int_V \int_{\Gamma_+} f_{n+1}^p\, |\mathbf{n}\mathbf{v}|\, d\sigma\, d\mathbf{v}\, d\tau \\ &\quad + \int_0^t \int_D \int_V L f_{n+1}^p\, d\mathbf{x}\, d\mathbf{v}\, d\tau \\ &= \int_D \int_V F_0^p\, d\mathbf{x}\, d\mathbf{v} + \int_0^t \int_V \int_{\Gamma_-} f_{n+1}^p\, |\mathbf{n}\mathbf{v}|\, d\sigma\, d\mathbf{v}\, d\tau \\ &\quad + \int_0^t \int_D \int_V \int_V K f_n^p\, d\mathbf{x}\, d\mathbf{v}'\, d\mathbf{v}\, d\tau \end{aligned}$$

Here, using (1.32) and the dominated convergence theorem

$$\begin{aligned} &\lim_{n \to \infty} \int_0^t \int_V \int_{\Gamma_+} f_{n+1}^p\, |\mathbf{n}\mathbf{v}|\, d\sigma\, d\mathbf{v}\, d\tau \\ &\quad = \int_0^t \int_V \int_{\Gamma_+} f^p\, |\mathbf{n}\mathbf{v}|\, d\sigma\, d\mathbf{v}\, d\tau \\ &\quad = \lim_{n \to \infty} \int_0^t \int_V \int_{\Gamma_-} f_{n+1}^p\, |\mathbf{n}\mathbf{v}|\, d\sigma\, d\mathbf{v}\, d\tau \end{aligned}$$

and

$$\lim_{n\to\infty}\int_0^t\int_D\int_V Lf_{n+1}^p\,d\mathbf{x}\,d\mathbf{v}\,d\tau$$
$$=\int_0^t\int_D\int_V Lf^p\,d\mathbf{x}\,d\mathbf{v}\,d\tau$$
$$=\lim_{n\to\infty}\int_0^t\int_D\int_V\int_V Kf_n^p\,d\mathbf{x}\,d\mathbf{v}\,d\mathbf{v}'\,d\tau$$

Then

$$\int_D\int_V f^p\,d\mathbf{x}\,d\mathbf{v}=\lim_{n\to\infty}\int_D\int_V f_{n+1}^p\,d\mathbf{x}\,d\mathbf{v}=\int_D\int_V F_0^p\,d\mathbf{x}\,d\mathbf{v},\qquad p\in\mathbb{N}$$

Using (1.35) and the monotone convergence theorem, the mass conservation relation (1.33) follows, when $p\to\infty$, so statement A is proved.

(B) Compare the function $\bar{f}=\bar{f}(\mathbf{x},\mathbf{v},t)$ with the iterate functions $f_n=f_n(\mathbf{x},\mathbf{v},t)$ defined in (1.14). By induction, using (1.10) and (1.14), one finds that

$$f_n(\mathbf{x},\mathbf{v},t)\leqslant\bar{f}(\mathbf{x},\mathbf{v},t),\qquad \mathbf{x}\in D,\quad \mathbf{v}\in V,\quad t\in\mathbb{R}_+,\quad n\in\mathbb{N}$$

so

$$f(\mathbf{x},\mathbf{v},t)=\lim_{n\to\infty}f_n(\mathbf{x},\mathbf{v},t)\leqslant\bar{f}(\mathbf{x},\mathbf{v},t)$$

and

$$f(\mathbf{x},\mathbf{v},t)-\bar{f}(\mathbf{x},\mathbf{v},t)\leqslant 0,\qquad \mathbf{x}\in D,\quad \mathbf{v}\in V,\quad t\in\mathbb{R}_+$$

But, by A and (1.34),

$$\int_D\int_V f\,d\mathbf{x}\,d\mathbf{v}=\int_D\int_V F_0\,d\mathbf{x}\,d\mathbf{v}\geqslant\int_D\int_V\bar{f}\,d\mathbf{x}\,d\mathbf{v}$$

so

$$\int_D\int_V(f-\bar{f})\,d\mathbf{x}\,d\mathbf{v}\geqslant 0$$

Then for $t>0$

$$\bar{f}(\cdot,\cdot,t)=f(\cdot,\cdot,t)\qquad\text{a.e. in } D\times V \quad ∎$$

Remark. The mass conservation and uniqueness results in Theorem 1.4, which are improvements of corresponding results in ref. 11, can also be obtained without detailed balance assumptions, using other types of assumptions on the functions B and ψ in the collision term (cf. ref. 5 and also Section 3).

The results in Theorem 1.4 are also easily obtained if (1.18) holds and $f(\mathbf{x}, \mathbf{v}, t)\, |\mathbf{nv}| \in L^1(\Gamma \times V \times [0, T])$; let $n \to \infty$ in the Green's identity (1.21).

2. THE *H*-FUNCTIONAL IN THE CUTOFF CASE

An H-functional $H_E(f)$, which is a (negative) entropy functional, can be defined by

$$H_E(f)(t) = \int_{D^0} \int_V f(\mathbf{x}, \mathbf{v}, t) \log[f(\mathbf{x}, \mathbf{v}, t)/E(\mathbf{v})] \, d\mathbf{x} \, d\mathbf{v} \qquad (2.1)$$

where $D^0 = D \backslash \Gamma$, D compact, and $E(\mathbf{v})$ is a given function, (see, e.g., refs. 16 and 17). As usual, we define $0 \log 0 = 0$. Analogously, we also define an (integrated) relative H-functional for the boundary $\Gamma = \partial D$ (cf. ref. 4, p. 138),

$$H_E^b(f)(t) = \int_0^t \int_\Gamma \int_V f(\mathbf{x}, \mathbf{v}, \tau) \log[f(\mathbf{x}, \mathbf{v}, \tau)/E(\mathbf{v})](\mathbf{nv}) \, d\sigma \, d\mathbf{v} \, d\tau \qquad (2.2)$$

where $\mathbf{nv} > 0$ on Γ_+ and $\mathbf{nv} < 0$ on Γ_-.

The main result of this section is given in Theorem 2.1, which is an H-theorem for our solution f to the linear Boltzmann equation under detailed balance and general boundary conditions. From this theorem it follows that the H-functional (2.1) for our solution is nonincreasing in time,

$$H_E(f)(t) \leqslant H_E(F_0), \qquad t \in \mathbb{R}_+ \qquad (2.3)$$

Such H-theorems, usually formulated as in (2.3), have been proved in various situations; for instance, by Voigt[16] for linear operators, by Arkeryd[1] for the nonlinear, space-homogeneous Boltzmann equation, and by Cercignani[4] including the boundary, generalizing a boundary H-theorem by Darrozes-Guiraud. The proof of our theorem combines the methods in refs. 1, 4, and 16.

Theorem 2.1. Let $f = f(\mathbf{x}, \mathbf{v}, t)$ be the mild solution of problem (0.1)–(0.3) given in Theorem 1.3, and let the detailed balance relations (1.27) and (1.30) with (1.31) hold, together with (1.32). If $H_E(F_0)$ exists, then the relative H-functional $H_E(f)(t)$ in (2.1) exists for $t > 0$, and it is nonincreasing in time. Moreover,

$$H_E(f)(t) - H_E(F_0) \leqslant \int_0^t N_E(f)(\tau) \, d\tau + \int_0^t N_E^b(f)(\tau) \, d\tau \qquad (2.4)$$

where

$$N_E(f)(t)=\frac{1}{2}\int_D\int_V\int_V K(\mathbf{x},\mathbf{v}\to\mathbf{v}')\,E(\mathbf{v})$$
$$\times\left[\frac{f(\mathbf{x},\mathbf{v}',t)}{E(\mathbf{v}')}-\frac{f(\mathbf{x},\mathbf{v},t)}{E(\mathbf{v})}\right]$$
$$\times\left[\log\frac{f(\mathbf{x},\mathbf{v},t)}{E(\mathbf{v})}-\log\frac{f(\mathbf{x},\mathbf{v}',t)}{E(\mathbf{v}')}\right]d\mathbf{x}\,d\mathbf{v}\,d\mathbf{v}' \tag{2.5}$$

and

$$N_E^b(f)(t)=\frac{1}{2}\int_\Gamma\iint_{\substack{\mathbf{nv}<0\\ \mathbf{nv}'>0}} R(\mathbf{x},\mathbf{v}'\to\mathbf{v})\,|\mathbf{nv}'|\,E(\mathbf{v}')$$
$$\times\left[\frac{f(\mathbf{x},\mathbf{v}',t)}{E(\mathbf{v}')}-\frac{f(\mathbf{x},\mathbf{v},t)}{E(\mathbf{v})}\right]$$
$$\times\left[\log\frac{f(\mathbf{x},\mathbf{v},t)}{E(\mathbf{v})}-\log\frac{f(\mathbf{x},\mathbf{v}',t)}{E(\mathbf{v}')}\right]d\sigma\,d\mathbf{v}\,d\mathbf{v}' \tag{2.6}$$

Proof. We shall use the elementary inequality

$$z\log z\geqslant z-1,\qquad \text{for } z>0$$

with equality if and only if $z=1$. With $z=f/E$, one finds that

$$f\log(f/E)\geqslant f-E \tag{2.7}$$

with equality in (2.7) if and only if $f=E$. With

$$f\log(f/E)=f\log^+(f/E)-f\log^-(f/E)$$

this gives

$$0\leqslant f\log^-(f/E)\leqslant f\log^+(f/E)-f+E \tag{2.8}$$

Therefore

$$f\log^-(f/E)\in L_+^1(D\times V)$$

if

$$f\log^+(f/E)\in L_+^1(D\times V)\qquad\text{and}\qquad E\in L_+^1(D\times V)$$

To prove Theorem 2.1, we start with a special case having a (double) change in the initial value function. Let, for $p, j = 1, 2, 3,...,$

$$F_0^{p,j}(\mathbf{x}, \mathbf{v}) = F_0^p(\mathbf{x}, \mathbf{v}) + \frac{1}{j} E(\mathbf{v}), \qquad \mathbf{x} \in D, \quad \mathbf{v} \in V \tag{2.9a}$$

where

$$F_0^p(\mathbf{x}, \mathbf{v}) = \min(F_0(\mathbf{x}, \mathbf{v}), pE(\mathbf{v})) \tag{2.9b}$$

If $f^{p,j} = f^{p,j}(\mathbf{x}, \mathbf{v}, t)$ is the mild solution from Theorem 1.3 with initial value $F_0^{p,j}$, then equality holds in (2.4), i.e.,

$$H_E(f^{p,j})(t) - H_E(F_0^{p,j}) = \int_0^t N_E(f^{p,j})(\tau)\, d\tau + \int_0^t N_E^b(f^{p,j})(\tau)\, d\tau \tag{2.10}$$

To prove this *statement*, we start with the iterates f_n^p, $n = 0, 1, 2,...,$ and $f^p = \lim_{n \to \infty} f_n^p$, using the initial value function F_0^p. Now let

$$f_n^{p,j}(\mathbf{x}, \mathbf{v}, t) = f_n^p(\mathbf{x}, \mathbf{v}, t) + \frac{1}{j} E(\mathbf{v}) \tag{2.11a}$$

Then the increasing sequence $\{f_n^{p,j}\}_{n=0}^{\infty}$ also converges pointwise (when $n \to \infty$) to a function

$$f^{p,j}(\mathbf{x}, \mathbf{v}, t) = f^p(\mathbf{x}, \mathbf{v}, t) + \frac{1}{j} E(\mathbf{v}) \tag{2.11b}$$

which by linearity is a mild solution corresponding to the initial value function $F_0^{p,j}$.

Now we get from (1.20) in Section 1, using differentiation along the characteristics, that, for a.e. $(\mathbf{x}, \mathbf{v}, t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times [0, T]$,

$$\frac{d}{dt}(f_{n+1}^{p,j}(\mathbf{x} + t\mathbf{v}, \mathbf{v}, t)) = \int_V K(\mathbf{x} + t\mathbf{v}, \mathbf{v}' \to \mathbf{v})\, f_n^{p,j}(\mathbf{x} + t\mathbf{v}, \mathbf{v}', t)\, d\mathbf{v}$$
$$- L(\mathbf{x} + t\mathbf{v}, \mathbf{v})\, f_{n+1}^{p,j}(\mathbf{x} + t\mathbf{v}, \mathbf{v}, t)$$

Multiplying this equation with $\{1 + \log[f_{n+1}^{p,j}(\mathbf{x} + t\mathbf{v}, \mathbf{v}, t)/E(\mathbf{v})]\}$ and using (1.2), one has

$$\frac{d}{dt}\left[f_{n+1}^{p,j}(\mathbf{x} + t\mathbf{v}, \mathbf{v}, t) \log\left(\frac{f_{n+1}^{p,j}(\mathbf{x} + t\mathbf{v}, \mathbf{v}, t)}{E(\mathbf{v})}\right)\right]$$
$$= \int_V [K(\mathbf{x} + t\mathbf{v}, \mathbf{v}' \to \mathbf{v})\, f_n^{p,j}(\mathbf{x} + t\mathbf{v}, \mathbf{v}', t)$$
$$- K(\mathbf{x} + t\mathbf{v}, \mathbf{v} \to \mathbf{v}')\, f_{n+1}^{p,j}(\mathbf{x} + t\mathbf{v}, \mathbf{v}, t)]$$
$$\times \left[1 + \log\left(\frac{f_{n+1}^{p,j}(\mathbf{x} + t\mathbf{v}, \mathbf{v}, t)}{E(\mathbf{v})}\right)\right] d\mathbf{v}'$$

a.e. $(\mathbf{x}, \mathbf{v}, t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times [0, T]$, $T > 0$.

Then, integrating

$$\int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \cdots d\mathbf{x}\, d\mathbf{v}\, d\tau$$

and using a Green's identity, we get (after a change of variables $\mathbf{x}+\tau\mathbf{v}\mapsto\mathbf{x}$), with $D^0 = D\backslash\Gamma$,

$$\begin{aligned}
&\int_{D^0}\int_V f_{n+1}^{p,j}(\mathbf{x},\mathbf{v},t)\log[f_{n+1}^{p,j}(\mathbf{x},\mathbf{v},t)/E(\mathbf{v})]\,d\mathbf{x}\,d\mathbf{v}\\
&\qquad -\int_{D^0}\int_V F_0^{p,j}(\mathbf{x},\mathbf{v})\log[F_0^{p,j}(\mathbf{x},\mathbf{v})/E(\mathbf{v})]\,d\mathbf{x}\,d\mathbf{v}\\
&\qquad +\int_0^t\int_\Gamma\int_{\mathbf{nv}>0} f_{n+1}^{p,j}(\mathbf{x},\mathbf{v},\tau)\log[f_{n+1}^{p,j}(\mathbf{x},\mathbf{v},\tau)/E(\mathbf{v})]\,|\mathbf{nv}|\,d\sigma\,d\mathbf{v}\,d\tau\\
&\qquad -\int_0^t\int_\Gamma\int_{\mathbf{nv}<0} f_{n+1}^{p,j}(\mathbf{x},\mathbf{v},\tau)\log[f_{n+1}^{p,j}(\mathbf{x},\mathbf{v},\tau)/E(\mathbf{v})]\,|\mathbf{nv}|\,d\sigma\,d\mathbf{v}\,d\tau\\
&\quad =\int_0^t\int_D\int_V\int_V [K(\mathbf{x},\mathbf{v}'\to\mathbf{v})\,f_n^{p,j}(\mathbf{x},\mathbf{v}',\tau)-K(\mathbf{x},\mathbf{v}\to\mathbf{v}')\,f_{n+1}^{p,j}(\mathbf{x},\mathbf{v},\tau)]\\
&\qquad \times\{1+\log[f_{n+1}^{p,j}(\mathbf{x},\mathbf{v},\tau)/E(\mathbf{v})]\}\,d\mathbf{x}\,d\mathbf{v}\,d\mathbf{v}'\,d\tau
\end{aligned}$$

Here

$$1/j \leqslant f_{n+1}^{p,j}(\mathbf{x},\mathbf{v},t)/E(\mathbf{v}) \leqslant p+1$$

so

$$|\log(f_{n+1}^{p,j}/E)| \leqslant \log j + \log(p+1)$$

for all $n\in N$. Using this inequality together with the assumptions (1.32) and the dominated convergence theorem, when $n\to\infty$, we arrive at the following equation for $f^{p,j}=\lim_{n\to\infty} f_n^{p,j}$ (with the boundary integrals on the right-hand side):

$$\begin{aligned}
&\int_{D^0}\int_V f^{p,j}(\mathbf{x},\mathbf{v},t)\log[f^{p,j}(\mathbf{x},\mathbf{v},t)/E(\mathbf{v})]\,d\mathbf{x}\,d\mathbf{v}\\
&\qquad -\int_{D^0}\int_V F_0^{p,j}(\mathbf{x},\mathbf{v})\log[F_0^{p,j}(\mathbf{x},\mathbf{v})/E(\mathbf{v})]\,d\mathbf{x}\,d\mathbf{v}\\
&\quad =\int_0^t\int_D\int_V\int_V [K(\mathbf{x},\mathbf{v}'\to\mathbf{v})\,f^{p,j}(\mathbf{x},\mathbf{v}',\tau)-K(\mathbf{x},\mathbf{v}\to\mathbf{v}')\,f^{p,j}(\dot{\mathbf{x}},\mathbf{v},\tau)]\\
&\qquad \times\{1+\log[f^{p,j}(\mathbf{x},\mathbf{v},\tau)/E(\mathbf{v})]\}\,d\mathbf{x}\,d\mathbf{v}\,d\mathbf{v}'\,d\tau\\
&\qquad -\int_0^t\int_\Gamma\int_V f^{p,j}(\mathbf{x},\mathbf{v},\tau)\log[f^{p,j}(\mathbf{x},\mathbf{v},\tau)/E(\mathbf{v})]\cdot(\mathbf{nv})\,d\sigma\,d\mathbf{v}\,d\tau \qquad (2.12)
\end{aligned}$$

Here the first term, i.e., the collision integral, on the right-hand side can be written, after a change of variables $\mathbf{v} \mapsto \mathbf{v}'$, $\mathbf{v}' \mapsto \mathbf{v}$ and using the detailed balance relation (1.27),

$$\frac{1}{2}\int_0^t \int_D \int_V \int_V [K(\mathbf{x}, \mathbf{v}' \to \mathbf{v})\, f^{p,j}(\mathbf{x}, \mathbf{v}', \tau) - K(\mathbf{x}, \mathbf{v} \to \mathbf{v}')\, f^{p,j}(\mathbf{x}, \mathbf{v}, \tau)]$$

$$\times \{\log[f^{p,j}(\mathbf{x}, \mathbf{v}, \tau)/E(\mathbf{v})]$$

$$- \log[f^{p,j}(\mathbf{x}, \mathbf{v}', \tau)/E(\mathbf{v}')]\}\, d\mathbf{x}\, d\mathbf{v}\, d\mathbf{v}'\, d\tau$$

$$= \frac{1}{2}\int_0^t \int_D \int_V \int_V K(\mathbf{x}, \mathbf{v} \to \mathbf{v}')\, E(\mathbf{v})$$

$$\times [f^{p,j}(\mathbf{x}, \mathbf{v}', \tau)/E(\mathbf{v}') - f^{p,j}(\mathbf{x}, \mathbf{v}, \tau)/E(\mathbf{v})]$$

$$\times \{\log\, [f^{p,j}(\mathbf{x}, \mathbf{v}, \tau)/E(\mathbf{v})]$$

$$- \log[f^{p,j}(\mathbf{x}, \mathbf{v}', \tau)/E(\mathbf{v}')]\, d\mathbf{x}\, d\mathbf{v}\, d\mathbf{v}'\, d\tau$$

$$= \int_0^t N_E(f^{p,j})(\tau)\, d\tau$$

Furthermore, in the second term, the boundary integral, on the right-hand side of (2.12) we transform I^b as follows, writing $f(\mathbf{v})$ for $f^{p,j}(\mathbf{x}, \mathbf{v}, \tau)$ and using (0.3), (0.6):

$$I^b(\mathbf{x}, \tau) \equiv -\int_V f(\mathbf{v}) \log[f(\mathbf{v})/E(\mathbf{v})](\mathbf{nv})\, d\mathbf{v}$$

$$= \int_{\mathbf{nv} < 0} f(\mathbf{v}) \log[f(\mathbf{v})/E(\mathbf{v})]\, |\mathbf{nv}|\, d\mathbf{v}$$

$$- \int_{\mathbf{nv}' > 0} f(\mathbf{v}') \log[f(\mathbf{v}')/E(\mathbf{v}')]\, |\mathbf{nv}'|\, d\mathbf{v}'$$

$$= \int_{\mathbf{nv} < 0} \log[f(\mathbf{v})/E(\mathbf{v})] \int_{\mathbf{nv}' > 0} R(\mathbf{v}' \to \mathbf{v})\, |\mathbf{nv}'|\, f(\mathbf{v}')\, d\mathbf{v}'\, d\mathbf{v}$$

$$- \int_{\mathbf{nv} < 0} R(\mathbf{v}' \to \mathbf{v})\, d\mathbf{v} \int_{\mathbf{nv}' > 0} f(\mathbf{v}') \log[f(\mathbf{v}')/E(\mathbf{v}')]\, |\mathbf{nv}'|\, d\mathbf{v}'$$

$$= \iint_{\substack{\mathbf{nv} < 0\\ \mathbf{nv}' > 0}} R(\mathbf{v}' \to \mathbf{v})\, |\mathbf{nv}'|\, f(\mathbf{v}')\{\log[f(\mathbf{v})/E(\mathbf{v})]$$

$$- \log[f(\mathbf{v}')/E(\mathbf{v}')]\}\, d\mathbf{v}\, d\mathbf{v}'$$

Changing variables $\mathbf{v}' \mapsto -\mathbf{v}$, $\mathbf{v} \mapsto -\mathbf{v}'$ and using the detailed balance relation (1.30), we get

$$I^b = \iint_{\substack{\mathbf{n}(-\mathbf{v}')<0\\ \mathbf{n}(-\mathbf{v})>0}} R(-\mathbf{v} \to -\mathbf{v}')\, |\mathbf{nv}|\, f(-\mathbf{v}) \times \left[\log \frac{f(-\mathbf{v}')}{E(-\mathbf{v}')} - \log \frac{f(-\mathbf{v})}{E(-\mathbf{v})}\right] d\mathbf{v}\, d\mathbf{v}'$$

$$= \frac{1}{2} \iint_{\substack{\mathbf{nv}<0\\ \mathbf{nv}'>0}} R(\mathbf{v}' \to \mathbf{v})\, |\mathbf{nv}'|\, E(\mathbf{v}') \left[\frac{f(\mathbf{v})}{E(\mathbf{v})} - \frac{f(\mathbf{v}')}{E(\mathbf{v}')}\right] \times \left[\log \frac{f(\mathbf{v}')}{E(\mathbf{v}')} - \log \frac{f(\mathbf{v})}{E(\mathbf{v})}\right] d\mathbf{v}\, d\mathbf{v}'$$

Then we get, by integration [cf. (2.6)], that $\int_\Gamma I^b(\mathbf{x}, \tau)\, d\sigma = N_E^b(f^{p,j})(\tau)$, so proposition (2.10) holds.

Now, continuing the proof of Theorem 2.1, we will first let $j \to \infty$, and then $p \to \infty$. By the following inequalities, holding for all j,

$$\begin{aligned} &0 \leqslant f^{p,j} \log^+(f^{p,j}/E) \leqslant (p+1)\log(p+1) \cdot E \\ &0 \leqslant f^{p,j} \log^-(f^{p,j}/E) \leqslant [1 + (p+1)\log(p+1)] \cdot E \end{aligned} \tag{2.13}$$

and the dominated convergence theorem, it follows for $f^p = \lim_{j \to \infty} f^{p,j}$ that

$$\int_{D^0} \int_V f^p \log^\pm (f^p/E)\, d\mathbf{x}\, d\mathbf{v} = \lim_{j \to \infty} \left[\int_{D^0} \int_V f^{p,j} \log^\pm (f^{p,j}/E)\, d\mathbf{x}\, d\mathbf{v}\right]$$

so

$$H_E(f^p)(t) = \lim_{j \to \infty} H_E(f^{p,j})(t) \qquad \text{exists for } t > 0$$

Also

$$H_E(F_0^p) = \lim_{j \to \infty} H_E(F_0^{p,j}) \tag{2.14}$$

For $p \to \infty$, use Fatou's lemma with the nonnegative function S^p [cf. (2.7)],

$$S^p(\mathbf{x}, \mathbf{v}, t) \equiv f^p(\mathbf{x}, \mathbf{v}, t) \log[f^p(\mathbf{x}, \mathbf{v}, t)/E(\mathbf{v})] - f^p(\mathbf{x}, \mathbf{v}, t) + E(\mathbf{v}) \geqslant 0$$

Then

$$\int_{D^0} \int_V \lim_{p \to \infty} S^p(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v} \leqslant \lim_{p \to \infty} \inf \left[\int_{D^0} \int_V S^p(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v}\right]$$

so, with $f=\lim_{p\to\infty} f^p$,

$$\int_{D^0}\int_V [f\log(f/E)-f+E]\,d\mathbf{x}\,d\mathbf{v}$$

$$\leqslant \liminf_{p\to\infty}\int_{D^0}\int_V [f^p\log(f^p/E)-f^p+E]\,d\mathbf{x}\,d\mathbf{v}$$

$$=\liminf_{p\to\infty}\left[\int_{D^0}\int_V f^p\log(f^p/E)\,d\mathbf{x}\,d\mathbf{v}\right]$$

$$-\lim_{p\to\infty}\int_{D^0}\int_V f^p\,d\mathbf{x}\,d\mathbf{v}+\int_{D^0}\int_V E\,d\mathbf{x}\,d\mathbf{v}$$

where, by monotone convergence,

$$\lim_{p\to\infty}\int_{D^0}\int_V f^p\,d\mathbf{x}\,d\mathbf{v}=\int_{D^0}\int_V f\,d\mathbf{x}\,d\mathbf{v}$$

Then

$$\int_{D^0}\int_V f\log(f/E)\,d\mathbf{x}\,d\mathbf{v}\leqslant\liminf_{p\to\infty}\int_{D^0}\int_V f^p\log(f^p/E)\,d\mathbf{x}\,d\mathbf{v}$$

i.e.,

$$H_E(f)(t)\leqslant\liminf_{p\to\infty} H_E(f^p)(t),\qquad t\in\mathbb{R}_+ \tag{2.15}$$

with

$$f=f(\mathbf{x},\mathbf{v},t)=\lim_{p\to\infty} f^p(\mathbf{x},\mathbf{v},t)$$

Furthermore, by monotone and dominated convergence,

$$\lim_{p\to\infty} H_E(F_0^p)=H_E(F_0)<\infty \tag{2.16}$$

because

$$0\leqslant F_0^p\log^+(F_0^p/E)\leqslant F_0\log^+(F_0/E)$$
$$0\leqslant F_0^p\log^-(F_0^p/E)\leqslant F_0\log^+(F_0/E)+E$$

Now we also use Fatou's lemma for the two terms on the right-hand side of (2.10), when $p, j\to\infty$. These are integrals of nonpositive functions [cf. (2.5) and (2.6)], so we get

$$\limsup_{p,j\to\infty}\int_0^t N_E(f^{p,j})(\tau)\,d\tau \leqslant \int_0^t N_E(f)(\tau)\,d\tau$$
$$\limsup_{p,j\to\infty}\int_0^t N_E^b(f^{p,j})(\tau)\,d\tau \leqslant \int_0^t N_E^b(f)(\tau)\,d\tau \tag{2.17}$$

where $f=f(\mathbf{x},\mathbf{v},t)=\lim_{p,j\to\infty} f^{p,j}(\mathbf{x},\mathbf{v},t)$.

Summarizing, we find, by (2.10) and (2.14)–(2.17), that

$$\begin{aligned}
&H_E(f)(t)-H_E(F_0)\\
&\quad\leqslant \liminf_{p,j\to\infty} H_E(f^{p,j})(t)-\lim_{p,j\to\infty} H_E(F_0^{p,j})\\
&\quad\leqslant \limsup_{p,j\to\infty}\int_0^t N_E(f^{p,j})(\tau)\,d\tau+\limsup_{p,j\to\infty}\int_0^t N_E^b(f^{p,j})(\tau)\,d\tau\\
&\quad\leqslant \int_0^t N_E(f)(\tau)\,d\tau+\int_0^t N_E^b(f)(\tau)\,d\tau\leqslant 0
\end{aligned}$$

and the theorem follows. ∎

Remark. Using the notation (2.2) for the boundary term, we find that

$$H_E^b(f)(t)=-\int_0^t N_E^b(f)(\tau)\,d\tau$$

Then our H-theorem, Theorem 2.1, can be written as follows[4]:

$$H_E(f)(t)+H_E^b(f)(t)\leqslant H_E(F_0)+\int_0^t N_E(f)(\tau)\,d\tau$$

The results in this section can also be formulated in the following way.

Corollary 2.2. If the assumptions of Theorem 2.1 are satisfied for $t\geqslant 0$, then for $0\leqslant t_1\leqslant t_2$

$$H_E(f)(t_2)\leqslant H_E(f)(t_1)+\int_{t_1}^{t_2} N_E(f)(\tau)\,d\tau+\int_{t_1}^{t_2} N_E^b(f)(\tau)\,d\tau$$

i.e.,

$$H_E(f)(t_2)+H_E^b(f)(t_2)\leqslant H_E(f)(t_1)+H_E^b(f)(t_1)+\int_{t_1}^{t_2} N_E(f)(\tau)\,d\tau$$

3. ON HIGHER MOMENTS IN THE CUTOFF CASE

This section uses the form (0.9) for the collision term and studies some interactions including inverse kth power forces. First we get a theorem about local boundedness in time for higher moments of the solution from Section 1, under some assumptions, which include both soft and hard collision potentials. Then, under further assumptions, which include the case of hard potentials, we get a result about global boundedness in time for higher moments. The space domain D is here supposed to be compact (cf. Sections 0 and 5).

Notation. Given $q \geqslant 0$, $r > 1$, define the function $h_{q,r}\colon \mathbb{R}_+ \to \mathbb{R}_+$ as follows:

$$h_{q,r}(v) = \begin{cases} (1+v^2)^{q/2}, & 0 \leqslant v \leqslant r-1 \\ \text{const}, & r \leqslant v < \infty \end{cases} \tag{3.1}$$

and specify $h_{q,r}$ on $(r-1, r)$, so that its first-order derivative is continuous on $\mathbb{R}_+$ and decreasing on $(r-1, r)$. (Here we use $v = |\mathbf{v}|$, etc., for the absolute value of the velocity $\mathbf{v}$, etc.)

The following proposition holds for *local boundedness* in time for moments of our mild solution.

Proposition 3.1. Let $B(\theta, w)$ be continuous for $0 \leqslant \theta < \pi/2$, $w > 0$. Suppose there exist constants C_B and λ with $0 \leqslant \lambda < 2$, such that (for all $\mathbf{v}, \mathbf{v}_* \in V$)

$$\int_0^{\pi/2} w \cos\theta\, B(\theta, w)\, d\theta \leqslant C_B(1+w)^\lambda, \qquad w = |\mathbf{v} - \mathbf{v}_*| \tag{3.2}$$

Assume there exists a constant $C_{q_0,\lambda}$, such that

$$\int_V (1+\mathrm{v}_*)^{\lambda + \max(1,\, q_0 - 1)}\, \psi(\mathbf{x}, \mathbf{v}_*)\, d\mathbf{v}_* \leqslant C_{q_0,\lambda}, \qquad \mathbf{x} \in D \tag{3.3}$$

Suppose that the boundary function R in (0.3) satisfies

$$R(\mathbf{x}, \mathbf{v} \to \mathbf{v}') = 0, \qquad v' > v, \quad \mathbf{v}, \mathbf{v}' \in V, \quad \mathbf{x} \in \Gamma \tag{3.4}$$

(representing a "nonheating boundary"). If

$$(1+v^2)^{q_0/2}\, F_0(\mathbf{x}, \mathbf{v}) \in L^1_+(D \times V)$$

then the mild solution f (given in Theorem 1.3) to the linear Boltzmann equation with (general) boundary conditions satisfies

$$\int_D \int_V (1+v^2)^{q/2} f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v} \leqslant e^{At} \int_D \int_V (1+v^2)^{q/2}\, F_0(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v} \tag{3.5}$$

for $0 < q \leqslant q_0$, some constant A, and all $t > 0$.

Proof. Start with the iterates $\{f_n\}_{n=0}^{\infty}$, defining $f = \lim_{n \to \infty} f_n$, and use [cf. (1.20)]

$$\frac{d}{dt}[f_n(\mathbf{x}+t\mathbf{v}, \mathbf{v}, t)] + L(\mathbf{x}+t\mathbf{v}, \mathbf{v}) f_n(\mathbf{x}+t\mathbf{v}, \mathbf{v}, t) = \int_V K(\mathbf{x}+t\mathbf{v}, \mathbf{v}' \to \mathbf{v}) f_{n-1}(\mathbf{x}+t\mathbf{v}, \mathbf{v}', t)\, d\mathbf{v}' \tag{3.6}$$

where all terms belong to $L^1(\mathbb{R}^3 \times \mathbb{R}^3 \times [0, T])$, $T > 0$ (cf. Section 1). Then, using Lemma 1.2 and multiplying (3.6) with the bounded function $h_{q,r}(v)$ [cf. (3.1)], we get

$$\frac{d}{dt}[h_{q,r}(v) f_n(\mathbf{x}+t\mathbf{v}, \mathbf{v}, t)] \leqslant \int_V h_{q,r}(v) K(\mathbf{x}+t\mathbf{v}, \mathbf{v}' \to \mathbf{v}) f_n(\mathbf{x}+t\mathbf{v}, \mathbf{v}', t)\, d\mathbf{v}' - h_{q,r}(v) L(\mathbf{x}+t\mathbf{v}, \mathbf{v}) f_n(\mathbf{x}+t\mathbf{v}, \mathbf{v}, t)$$

Integration over $\mathbb{R}^3 \times \mathbb{R}^3 \times [0, T]$, using a Green's identity and a change of variables, gives.

$$\int_D \int_V h_{q,r}(v) f_n(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v} - \int_D \int_V h_{q,r}(v) F_0(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v} + \int_0^t \int_\Gamma \int_V h_{q,r}(v) f_n(\mathbf{x}, \mathbf{v}, \tau)(\mathbf{nv})\, d\sigma\, d\mathbf{v}\, d\tau$$
$$\leqslant \int_0^t \int_D \int_V \int_V [h_{q,r}(v') - h_{q,r}(v)]\, K(\mathbf{x}, \mathbf{v}, \to \mathbf{v}') f_n(\mathbf{x}, \mathbf{v}, \tau)\, d\mathbf{x}\, d\mathbf{v}\, d\mathbf{v}'\, d\tau \tag{3.7}$$

after some further changes of variables and use of (1.2) for the collision term on the right-hand side. The collision integral in (3.7) can also be written

$$\int_0^t \int_D \int_V \int_V \int_S [h_{q,r}(v') - h_{q,r}(v)]\, \psi(\mathbf{x}, \mathbf{v}_*)\, B(\theta, w) f_n(\mathbf{x}, \mathbf{v}, \tau)\, d\theta\, d\zeta\, d\mathbf{x}\, d\mathbf{v}\, d\mathbf{v}_*$$

Writing the boundary term in (3.7) on the right-hand side, we get, by (0.3), (0.6), and (3.4), that

$$-\int_0^t \int_\Gamma \int_V h_{q,r}(v) f_n(\mathbf{x}, \mathbf{v}, \tau)(\mathbf{nv})\, d\sigma\, d\mathbf{v}\, d\tau$$
$$= \int_0^t \int_\Gamma \iint_{\substack{\mathbf{nv}' < 0 \\ \mathbf{nv} > 0}} [h_{q,r}(v') - h_{q,r}(v)] \times R(\mathbf{x}, \mathbf{v} \to \mathbf{v}') f_n(\mathbf{x}, \mathbf{v}, \tau)\, |\mathbf{nv}|\, d\sigma\, d\mathbf{v}\, d\mathbf{v}'\, d\tau \leqslant 0$$

Then

$$\int_D \int_V h_{q,r}(v)\, f_n(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v} - \int_D \int_V h_{q,r}(v)\, F_0(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v}$$

$$\leqslant \int_0^t \int_D \int_V \int_V \int_S [h_{q,r}(v') - h_{q,r}(v)]\, \psi(\mathbf{x}, \mathbf{v}_*)$$

$$\times B(\theta, w)\, f_n(\mathbf{x}, \mathbf{v}, \tau)\, d\theta\, d\zeta\, d\mathbf{x}\, d\mathbf{v}\, d\mathbf{v}_*\, d\tau \tag{3.8}$$

where, for $v' > v$ (by the construction of $h_{q,r}$),

$$h_{q,r}(v') - h_{q,r}(v) \leqslant [1 + (v')^2]^{q/2} - (1 + v^2)^{q/2}$$

Here we use the following essential inequality, which holds for the velocities in a binary collision[12]:

$$[1 + (v')^2]^{q/2} - (1 + v^2)^{q/2} \leqslant qK_1 \cos\theta\, w(1 + v_*)^{\max(1, q-1)}\, (1 + v^2)^{(q-2)/2} \tag{3.9}$$

for some constant K_1.

Using this together with (3.2) and (3.3), we find that the right-hand side of (3.8) is less than

$$qK_1 \int_0^t \int_D \int_V \int_V \int_0^{2\pi} \int_0^{\pi/2} w \cos\theta\, B(\theta, w)(1 + v_*)^{\max(1, q-1)}$$

$$\times \psi(\mathbf{x}, \mathbf{v}_*)(1 + v^2)^{(q-2)/2}$$

$$\times f_n(\mathbf{x}, \mathbf{v}, \tau)\, d\mathbf{x}\, d\mathbf{v}\, d\mathbf{v}_*\, d\theta\, d\zeta\, d\tau$$

$$\leqslant 2\pi q K_1 C_B 2^\lambda \int_0^t \int_D \int_V \int_V (1 + v_*)^{\lambda + \max(1, q-1)}\, \psi(\mathbf{x}, \mathbf{v}_*)(1 + v^2)^{(q+\lambda-2)/2}$$

$$\times f_n(\mathbf{x}, \mathbf{v}, \tau)\, d\mathbf{x}\, d\mathbf{v}\, d\mathbf{v}_*\, d\tau$$

$$\leqslant 8\pi q K_1 C_B C_{q_0, \lambda} \int_0^t \int_D \int_V (1 + v^2)^{(q+\lambda-2)/2} f_n(\mathbf{x}, \mathbf{v}, \tau)\, d\mathbf{x}\, d\mathbf{v}\, d\tau$$

Here we also used that $1 + w \leqslant (1 + v_*) \cdot 2(1 + v^2)^{1/2}$.

Let $\delta = \min(q, 2 - \lambda) > 0$. Then, by (3.8), when $r \to \infty$,

$$\int_D \int_V (1 + v^2)^{q/2} f_n(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v}$$

$$\leqslant \int_D \int_V (1 + v^2)^{q/2}\, F_0(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v}$$

$$+ A \int_0^t \int_D \int_V (1 + v^2)^{(q-\delta)/2} f_n(\mathbf{x}, \mathbf{v}, \tau)\, d\mathbf{x}\, d\mathbf{v}\, d\tau$$

$$\leqslant \int_D \int_V (1+v^2)^{q/2} F_0(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v}$$

$$+ A \int_0^t \int_D \int_V (1+v^2)^{q/2} f_n(\mathbf{x}, \mathbf{v}, \tau)\, d\mathbf{x}\, d\mathbf{v}\, d\tau$$

with some constant A $(= 8\pi q K_1 C_B C_{q_0, \lambda})$. Now using a Gronwall lemma, we get

$$\int_D \int_V (1+v^2)^{q/2} f_n(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v} \leqslant e^{At} \int_D \int_V (1+v^2)^{q/2} F_0(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v}$$

Let $n \to \infty$ and use that $f_n \nearrow f$. Then (3.5) follows, giving local boundedness in time for soft and hard potentials, $0 \leqslant \lambda < 2$. ∎

Remark. The assumption (3.4) about "nonheating boundary" is satisfied, for instance, by the specular boundary condition.

The rest of this section is concerned with *global boundedness* in time for *hard* potentials in the collision term.

Theorem 3.2. Let the assumptions of Theorem 3.1 be satisfied with $1 \leqslant \lambda < 2$. Moreover, suppose there are constants $\bar{C}_B > 0$ and $C_0 > 0$ such that

$$\int_0^{\pi/2} w \cos^2\theta\, B(\theta, w)\, d\theta \geqslant \bar{C}_B w^{\lambda} \tag{3.10}$$

and

$$\int_V \psi(\mathbf{x}, \mathbf{v}_*)\, d\mathbf{v}_* \geqslant C_0, \qquad \mathbf{x} \in D \backslash \Gamma \tag{3.11}$$

Assume that the function $E(\mathbf{v})$ in the detailed balance relations (1.27) and (1.30) with (1.31) satisfies

$$(1+v^2)^{q_0/2} K(\mathbf{x}, \mathbf{v} \to \mathbf{v}')\, E(\mathbf{v}) \in L^1_+(D \times V \times V)$$

$$(1+v^2)^{q_0/2} R(\mathbf{x}, \mathbf{v} \to \mathbf{v}')\, |\mathbf{n}\mathbf{v}|\, E(\mathbf{v}) \in L^1_+(\Gamma \times V \times V)$$

Then the mild solution f (given in Theorem 1.3) satisfies

$$\int_D \int_V (1+v^2)^{q/2} f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v} \leqslant A_q \int_D \int_V (1+v^2)^{q/2} F_0(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v} \tag{3.13}$$

for $0 < q \leqslant q_0$, some constant A_q, and all $t > 0$.

Proof. Start with a cutoff in the initial value [cf. (1.35)]

$$F_0^p(\mathbf{x}, \mathbf{v}) = \min(F_0(\mathbf{x}, \mathbf{v}), p \cdot E(\mathbf{v})), \qquad \mathbf{x} \in D, \quad \mathbf{v} \in V, \quad p = 1, 2, 3, \dots \tag{3.14}$$

and construct (as in Section 1) the increasing sequence of iterates $\{f_n^p\}_{n=0}^{\infty}$ with limit

$$f^p(\mathbf{x}, \mathbf{v}, t) = \lim_{n \to \infty} f_n^p(\mathbf{x}, \mathbf{v}, t) \leqslant p \cdot E(\mathbf{v}) \tag{3.15}$$

Now we use for f_n^p Eq. (3.6), which, after a multiplication with the bounded function $h_{q,r}$ [cf. (3.1)] and integration along the characteristics, gives [cf. also (0.3), (0.6), (1.2), (1.14), and (3.7)]

$$\begin{aligned}
&\int_D \int_V h_{q,r}(v)\, f_n^p(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v} - \int_D \int_V h_{q,r}(v)\, F_0^p(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v} \\
&\quad = \int_0^t \int_D \int_V \int_V h_{q,r}(v')\, K(\mathbf{x}, \mathbf{v} \to \mathbf{v}')\, f_{n-1}^p(\mathbf{x}, \mathbf{v}, \tau)\, d\mathbf{x}\, d\mathbf{v}\, d\mathbf{v}'\, d\tau \\
&\qquad - \int_0^t \int_D \int_V \int_V h_{q,r}(v)\, K(\mathbf{x}, \mathbf{v} \to \mathbf{v}')\, f_n^p(\mathbf{x}, \mathbf{v}, \tau)\, d\mathbf{x}\, d\mathbf{v}\, d\mathbf{v}'\, d\tau \\
&\qquad + \int_0^t \int_\Gamma \iint_{\substack{\mathbf{n}\mathbf{v}' < 0 \\ \mathbf{n}\mathbf{v} > 0}} h_{q,r}(v')\, R(\mathbf{x}, \mathbf{v} \to \mathbf{v}')\, f_{n-1}^p(\mathbf{x}, \mathbf{v}, \tau)\, |\mathbf{n}\mathbf{v}|\, d\sigma\, d\mathbf{v}\, d\mathbf{v}'\, d\tau \\
&\qquad - \int_0^t \int_\Gamma \iint_{\substack{\mathbf{n}\mathbf{v}' < 0 \\ \mathbf{n}\mathbf{v} > 0}} h_{q,r}(v)\, R(\mathbf{x}, \mathbf{v} \to \mathbf{v}')\, f_n^p(\mathbf{x}, \mathbf{v}, \tau)\, |\mathbf{n}\mathbf{v}|\, d\sigma\, d\mathbf{v}\, d\mathbf{v}'\, d\tau
\end{aligned}$$

Here, letting $n \to \infty$ and then $r \to \infty$, using (3.1), (3.13), (3.14), and the dominated convergence theorem, we get the following proposition:

$$\begin{aligned}
&\int_D \int_V (1 + v^2)^{q/2} f^p(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v} \\
&\quad - \int_D \int_V (1 + v^2)^{q/2} F_0^p(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v} \\
&= \int_0^t \int_D \int_V \int_V \{[1 + (v')^2]^{q/2} - (1 + v^2)^{q/2}\} \\
&\quad \times K(\mathbf{x}, \mathbf{v} \to \mathbf{v}')\, f^p(\mathbf{x}, \mathbf{v}, \tau)\, d\mathbf{x}\, d\mathbf{v}\, d\mathbf{v}'\, d\tau \\
&\quad + \int_0^t \int_\Gamma \iint_{\substack{\mathbf{n}\mathbf{v}' < 0 \\ \mathbf{n}\mathbf{v} > 0}} \{[1 + (\mathbf{v}')^2]^{q/2} - (1 + v^2)^{q/2}\} \\
&\quad \times R(\mathbf{x}, \mathbf{v} \to \mathbf{v}')\, f^p(\mathbf{x}, \mathbf{v}, \tau)\, |\mathbf{n}\mathbf{v}|\, d\sigma\, d\mathbf{v}\, d\mathbf{v}'\, d\tau
\end{aligned} \tag{3.16}$$

Taking the derivative with respect to t and using (3.4) gives

$$\frac{d}{dt}\left[\int_D \int_V (1+v^2)^{q/2} f^p(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v}\right]$$

$$\leqslant \int_D \int_V \int_V \{[1+(v')^2]^{q/2} - (1+v^2)^{q/2}\}$$

$$\times K(\mathbf{x}, \mathbf{v} \to \mathbf{v}')\, f^p(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v}\, d\mathbf{v}'$$

$$= \int_D \int_V \int_V \int_S \{[1+(v')^2]^{q/2} - (1+v^2)^{q/2}\}$$

$$\times B(\theta, w)\, \psi(\mathbf{x}, \mathbf{v}_*)\, f^p(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v}\, d\mathbf{v}_*\, d\theta\, d\zeta \qquad (3.17)$$

Here we use the following essential inequality for the velocities in a binary collision (cf. Proposition 1.2 in ref. 12):

$$[1+(v')^2]^{q/2} - (1+v^2)^{q/2}$$

$$\leqslant K_1 w \cos\theta\, (1+v_*)^{\max(1, q-1)}\, (1+v^2)^{(q-2)/2}$$

$$- K_2 w \cos^2\theta\, (1+v^2)^{(q-1)/2} \qquad (3.18)$$

with some positive constants K_1, $K_2 > 0$. Then it follows, by (3.2)–(3.4), (3.10)–(3.12), (3.17), and (3.18) and the elementary inequalities[12]

$$1 + w \leqslant 2(1+v_*)(1+v^2)^{1/2}$$

$$-w^\lambda \leqslant (1+v_*)^\lambda - 2^{1-\lambda}(1+v^2)^{\lambda/2}, \qquad 1 \leqslant \lambda < 2$$

that

$$\frac{d}{dt}\left[\int_D \int_V (1+v^2)^{q/2} f^p(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v}\right]$$

$$\leqslant K_1 \int_D \int_V \int_V \int_0^{2\pi} \int_0^{\pi/2} w \cos\theta\, B(\theta, w)(1+v_*)^{\max(1, q-1)}\, \psi(\mathbf{x}, \mathbf{v}_*)$$

$$\times (1+v^2)^{(q-2)/2} f^p(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v}\, d\mathbf{v}_*\, d\theta\, d\zeta$$

$$- K_2 \int_D \int_V \int_V \int_0^{2\pi} \int_0^{\pi/2} w \cos^2\theta\, B(\theta, w)\, \psi(\mathbf{x}, \mathbf{v}_*)$$

$$\times (1+v^2)^{(q-1)/2} f^p(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v}\, d\mathbf{v}_*\, d\theta\, d\zeta$$

$$\leqslant 2\pi K_1 C_B \int_D \int_V \int_V (1+w)^\lambda (1+v_*)^{\max(1,q-1)} \psi(\mathbf{x},\mathbf{v}_*)$$

$$\times (1+v^2)^{(q-2)/2} f^p(\mathbf{x},\mathbf{v},t)\,d\mathbf{x}\,d\mathbf{v}\,d\mathbf{v}_*$$

$$-2\pi K_2 \bar{C}_B \int_D \int_V \int_V w^\lambda \psi(\mathbf{x},\mathbf{v}_*)(1+v^2)^{(q-1)/2} f^p(\mathbf{x},\mathbf{v},t)\,d\mathbf{x}\,d\mathbf{v}\,d\mathbf{v}_*$$

$$\leqslant 2\pi K_1 C_B 2^\lambda \int_D \int_V \int_V (1+v_*)^{\lambda+\max(1,q-1)}$$

$$\times \psi(\mathbf{x},\mathbf{v}_*)(1+v^2)^{(q+\lambda-2)/2} f^p(\mathbf{x},\mathbf{v},t)\,d\mathbf{x}\,d\mathbf{v}\,d\mathbf{v}_*$$

$$+2\pi K_2 \bar{C}_B \int_D \int_V \int_V (1+v_*)^\lambda \psi(\mathbf{x},\mathbf{v}_*)(1+v^2)^{(q-1)/2} f^p(\mathbf{x},\mathbf{v},t)\,d\mathbf{x}\,d\mathbf{v}\,d\mathbf{v}_*$$

$$-2\pi K_2 \bar{C}_B 2^{1-\lambda} \int_D \int_V \int_V \psi(\mathbf{x},\mathbf{v}_*)(1+v^2)^{(q+\lambda-1)/2} f^p(\mathbf{x},\mathbf{v},t)\,d\mathbf{x}\,d\mathbf{v}\,d\mathbf{v}_*$$

$$\leqslant 8\pi K_1 C_B C_{q0,\lambda} \int_D \int_V (1+v^2)^{(q+\lambda-2)/2} f^p(\mathbf{x},\mathbf{v},t)\,d\mathbf{x}\,d\mathbf{v}$$

$$+2\pi K_2 \bar{C}_B C_{q_0,\lambda} \int_D \int_V (1+v^2)^{(q-1)/2} f^p(\mathbf{x},\mathbf{v},t)\,d\mathbf{x}\,d\mathbf{v}$$

$$-\pi K_2 \bar{C}_B C_0 \int_D \int_V (1+v^2)^{(q+\lambda-1)/2} f^p(\mathbf{x},\mathbf{v},t)\,d\mathbf{x}\,d\mathbf{v}$$

Let $\delta = \min(q, 2-\lambda)$ for $q>0$, $1 \leqslant \lambda < 2$, and let

$$M_q(t) = \int_D \int_V (1+v^2)^{q/2} f^p(\mathbf{x},\mathbf{v},t)\,d\mathbf{x}\,d\mathbf{v}, \qquad t \in \mathbb{R}_+ \tag{3.19}$$

Then it follows that

$$M'_q(t) \leqslant a_1 M_{q-\delta}(t) - a_0 M_q(t) \tag{3.20}$$

with positive constants $a_1, a_0 > 0$. Multiplying (3.20) with $e^{a_0 t}$ and integrating gives

$$e^{a_0 t} M_q(t) - M_q(0) \leqslant a_1 \int_0^t M_{q-\delta}(s)\, e^{a_0 s}\, ds$$

So, if $\sup_{t \in \mathbb{R}_+} M_{q-\delta}(t) \leqslant \overline{M_{q-\delta}}$, then

$$M_q(t) \leqslant M_q(0)\, e^{-a_0 t} + a_1 \overline{M_{q-\delta}} \int_0^t e^{(s-t)a_0}\, ds$$

$$\leqslant M_q(0) + (a_1/a_0)\, \overline{M_{q-\delta}} \tag{3.21}$$

But $M_0(t)$ is globally bounded [cf. (1.17)], so recursively we find by (3.21) that $M_\delta(t)$, $M_{2\delta}(t)$,..., $M_q(t)$ all are globally bounded in time. Furthermore, if $M_{q-\delta}(t) \leqslant A_{q-\delta} M_{q-\delta}(0)$, then by (3.19) and (3.21)

$$M_q(t) \leqslant M_q(0) + \frac{a_1}{a_0} A_{q-\delta} M_{q-\delta}(0) \leqslant \left(1 + \frac{a_1}{a_0} A_{q-\delta}\right) M_q(0)$$

so (3.13) holds for $f = f^p$.

Finally, letting $p \to \infty$, using $F_0^p \nearrow F_0$, $f^p \nearrow f$, then the result (3.13) follows for $f = f(\mathbf{x}, \mathbf{v}, t)$.

Remark. The results in Theorem 3.2, giving global boundedness in time for higher moments, hold (among others) for inverse kth power forces with $\lambda = \gamma + 1$, where $\gamma = (k-5)/(k-1)$, if $k \geqslant 5$ [cf. (0.10)], and with specular reflection at the boundary.

4. L^1-SOLUTIONS IN THE CASE OF INFINITE-RANGE FORCES WITHOUT CUTOFF

In this section the linear Boltzmann equation is considered without cutoff in the collision term, i.e., including infinite-range forces, and written in the following integral form, which can formally be derived from Eq. (0.1) with (0.2), (0.3), and (0.9)[11]:

$$\begin{aligned}
&\int_D \int_V g(\mathbf{x}, \mathbf{v}, t)\, f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v} \\
&\quad = \int_D \int_V g(\mathbf{x}, \mathbf{v}, 0)\, F_0(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v} \\
&\quad + \int_0^t \int_D \int_V \left[\mathbf{v} \cdot \operatorname{grad}_{\mathbf{x}} g(\mathbf{x}, \mathbf{v}, s) + \frac{\partial}{\partial s} g(\mathbf{x}, \mathbf{v}, s) \right] f(\mathbf{x}, \mathbf{v}, s)\, d\mathbf{x}\, d\mathbf{v}\, ds \\
&\quad + \int_0^t \int_D \int_V \int_V \int_S [g(\mathbf{x}, \mathbf{v}', s) - g(\mathbf{x}, \mathbf{v}, s)]\, \psi(\mathbf{x}, \mathbf{v}_*) \\
&\quad \times B(\theta, w)\, f(\mathbf{x}, \mathbf{v}, s)\, d\theta\, d\zeta\, d\mathbf{v}_*\, d\mathbf{x}\, d\mathbf{v}\, ds
\end{aligned} \tag{4.1}$$

for all test functions $g \in C_0^{1,\infty}$. Here

$$C_0^{1,\infty} = \{ g \in C^{1,\infty} \colon g(\mathbf{x}, \mathbf{v}, t) = 0,\ \mathbf{x} \in \Gamma = \partial D \}$$

where

$$\begin{aligned}
C^{1,\infty} = \{ g \in C^1(D \times V \times [0, \infty)) \colon \|g\|_1 &= \sup |g(\mathbf{x}, \mathbf{v}, t)| \\
&+ \sup \left| \frac{\partial}{\partial t} g(\mathbf{x}, \mathbf{v}, t) \right| + \sup |\operatorname{grad}_{\mathbf{x}} g(\mathbf{x}, \mathbf{v}, t)| \\
&+ \sup |\operatorname{grad}_{\mathbf{v}} g(\mathbf{x}, \mathbf{v}, t)| < \infty \}
\end{aligned}$$

The space domain D is here (as in Sections 1–3) supposed to be compact and (strictly) convex. The mathematical problems in the noncutoff case come from the nonintegrability of the function $B(\theta, w)$, when $\theta \to \pi/2-$ (cf. Section 0).

In refs. 11 and 12, Eq. (4.1) was considered for the periodic boundary case, with $d\mu$ instead of $f\,d\mathbf{x}\,d\mathbf{v}$, to get measure solutions $\mu(\mathbf{x}, \mathbf{v}, t)$. We will now use the H-theorem from Section 2 to get L^1-solutions of Eq. (4.1) by a method analogous to ref. 2 for the nonlinear, space-homogeneous case. In ref. 13, I used this method to get L^1-solutions to (4.1) in the periodic boundary case. For the given purpose I formulate the following essential lemma.

Lemma 4.1. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of $L^1_+(D \times V)$-functions, such that for some $\kappa_0 > 0$ there is a constant C_{κ_0},

$$\int_D \int_V (1+v)^{\kappa_0} f_n(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v} \leqslant C_{\kappa_0}, \qquad n \in \mathbb{N} \tag{4.2}$$

and such that for some function $E(\mathbf{v}) > 0$ with $[\log E(\mathbf{v})]/(1+v)^{\bar{\kappa}} \in L^\infty(V)$ for some $\bar{\kappa} < \kappa_0$, there is a constant C_E,

$$\int_D \int_V f_n(\mathbf{x}, \mathbf{v}) \log[f_n(\mathbf{x}, \mathbf{v})/E(\mathbf{v})]\, d\mathbf{x}\, d\mathbf{v} \leqslant C_E, \qquad n \in \mathbb{N} \tag{4.3}$$

Then the sequence $\{f_n\}_1^{\infty}$ contains a subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ converging weakly to a function $f \in L^1_+(D \times V)$, such that

$$\lim_{j\to\infty} \left[\int_D \int_V f_{n_j}(\mathbf{x}, \mathbf{v})\, g(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v}\right] = \int_D \int_V f(\mathbf{x}, \mathbf{v})\, g(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v} \tag{4.4}$$

if $g(\mathbf{x}, \mathbf{v})/(1+v)^{\kappa'} \in L^\infty(D \times V)$, $0 \leqslant \kappa' < \kappa_0$.

Proof. (Cf. Lemma 3.1 in ref. 1 or Lemma 2 in ref. 2.) The proof, which is based essentially on the well-known Dunford–Pettis' theorem, is analogous to that given by Arkeryd[1] and is omitted here. ∎

The main result of this section is the following theorem about the existence of L^1-solutions to Eq. (4.1). This theorem is an improvement of Theorems 2.1 and 2.2 in ref. 11 and Theorem 2.1 in ref. 12, which deal with the periodic boundary case.

Theorem 4.2. Let $B(\theta, w)$ be continuous for $0 \leqslant \theta < \pi/2$, $w > 0$. Suppose there exist constants C_B and λ, with $0 \leqslant \lambda < 2$, such that (for all $\mathbf{v}, \mathbf{v}_* \in V$)

$$\int_0^{\pi/2} w \cos\theta\, B(\theta, w)\, d\theta \leqslant C_B (1+w)^{\lambda}, \qquad w = |\mathbf{v} - \mathbf{v}_*| \tag{4.5}$$

and suppose

$$\int_0^{\pi/2} w \cos\theta\, B(\theta, w)\, d\theta = \mathcal{O}(1) \qquad \text{when } w \to 0 \tag{4.6}$$

Let the cutoff angle $\hat{\theta}(\hat{R}, w)$ increase with the cutoff radius $\hat{R}$, and let

$$\psi(\mathbf{x}, \mathbf{v}_*) = X(\mathbf{x})\, \Phi(\mathbf{v}_*) \tag{4.7}$$

with $X(\mathbf{x}) \in L_+^\infty(D)$ and $\Phi(\mathbf{v}_*)$ measurable on V. Assume there is a constant $C_{q_0,\lambda}$ such that

$$\int_V (1 + v_*)^{\lambda + \max(1, q_0 - 1)}\, \psi(\mathbf{x}, \mathbf{v}_*)\, d\mathbf{v}_* \leqslant C_{q_0,\lambda}, \qquad \mathbf{x} \in D \backslash \Gamma \tag{4.8}$$

$$(1 + v)^{q_0} F_0(\mathbf{x}, \mathbf{v}) \in L_+^1(D \times V) \qquad \text{for some} \quad q_0 > \lambda, \quad q_0 \geqslant 1 \tag{4.9}$$

Let

$$F_0(\mathbf{x}, \mathbf{v}) \log[F_0(\mathbf{x}, \mathbf{v})/E(\mathbf{v})] \in L^1(D \times V) \tag{4.10}$$

where $E(\mathbf{v}) > 0$, with

$$[\log E(\mathbf{v})]/(1 + v)^{\bar{q}} \in L^\infty(V) \qquad \text{for some} \quad \bar{q} < q_0$$

satisfy the detailed balance relations

$$\psi(\mathbf{x}, \mathbf{v}_*)\, E(\mathbf{v}) = \psi(\mathbf{x}, \mathbf{v}'_*)\, E(\mathbf{v}') \tag{4.11}$$

and

$$|\mathbf{nv}|\, R(\mathbf{x}, \mathbf{v} \to \mathbf{v}')\, E(\mathbf{v}) = |\mathbf{nv}'|\, R(\mathbf{x}, -\mathbf{v}' \to -\mathbf{v})\, E(-\mathbf{v}')$$

with

$$R(\mathbf{x}, \mathbf{v} \to \mathbf{v}') = 0, \qquad v' > v, \quad \mathbf{v}, \mathbf{v}', \in V \tag{4.12}$$

Then there exists (for all $t > 0$) a nonnegative solution $f = f(\mathbf{x}, \mathbf{v}, t) \in L_+^1(D \times V)$ to the linear, space-inhomogeneous Boltzmann equation in the integral form (4.1). The solution satisfies

$$\int_D \int_V f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v} = \int_D \int_V F_0(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v} \tag{4.13}$$

and

$$\int_D \int_V (1 + v^2)^{q/2} f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v} \leqslant e^{a_q t} \int_D \int_V (1 + v^2)^{q/2} F_0(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v} \tag{4.14}$$

for $0<q\leqslant q_0$ (with some constant a_q), and for $0\leqslant t\leqslant T<\infty$, where T is arbitrary.

Moreover, for $1\leqslant\lambda<2$, if there are constants $\bar{C}_B>0$ and $C_0>0$ such that

$$\int_0^{\pi/2} w\cos^2\theta\, B(\theta, w)\, d\theta\geqslant\bar{C}_B w^\lambda \tag{4.15}$$

and

$$\int_V \psi(\mathbf{x}, \mathbf{v}_*)\, d\mathbf{v}_*\geqslant C_0, \qquad \mathbf{x}\in D\backslash\Gamma \tag{4.16}$$

then

$$\int_D\int_V (1+v^2)^{q/2} f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v}\leqslant A_q\int_D\int_V (1+v^2)^{q/2} F_0(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v} \tag{4.17}$$

for $0<q\leqslant q_0$ (with some constant A_q), and for all $t\geqslant 0$.

Proof. (Cf. the proof of Theorem 3 in ref. 2.) Let $\{f^n(\mathbf{x}, \mathbf{v}, t)\}_{n=1}^\infty$ be a sequence of mild solutions to the linear Boltzmann equation (0.1) with (0.2), (0.3), and (0.9) with cutoff radius $\hat{R}=n$, $S=S_n$ (cf. Theorems 1.3 and 1.4). Then one finds, by straightforward calculations analogous to those in the previous section, that the functions $f^n=f^n(\mathbf{x}, \mathbf{v}, t)$ satisfy the integral equation (4.1). By the H-theorem, Theorem 2.1, and Lemma 4.1, we can select a subsequence $\{f^{n_j}\}_{j=1}^\infty$ converging weakly to a function $f\in L_+^1(D\times V)$ for all rational t, with $0\leqslant t\leqslant T<\infty$ if $0\leqslant\lambda<1$ and with $t\geqslant 0$ if $1\leqslant\lambda<2$ [cf. Theorems 3.1 and 3.2 to get (4.2)]. But for $g\in C_0^{1,\infty}$ the sequence $\int_D\int_V f^n(\mathbf{x}, \mathbf{v}, t)\, g(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v}$ is equicontinuous in t (cf. the proof of Theorem 2.1 in ref. 11), and then the subsequence $\{f^{n_j}\}$ converges weakly to a function $f\in L_+^1(D\times V)$ for all t (with $0\leqslant t\leqslant T<\infty$ and $t\geqslant 0$, respectively). One also finds that the function $f=w-\lim_{j\to\infty} f^{n_j}$ satisfies the integral equation (4.1) (cf. the proof of Theorem 2.1 in ref. 11 when $0\leqslant\lambda<1$, and Theorem 2.1 in ref. 12 when $1\leqslant\lambda<2$). Concerning local and global boundedness of higher moments, for $0\leqslant\lambda<2$ and $1\leqslant\lambda<2$, respectively, use Proposition 3.1 with inequality (3.5) and Theorem 3.2 with (3.13), which hold for f^{n_j} and then also for $f=w-\lim_{j\to\infty} f^{n_j}$. This completes the proof of Theorem 4.2. ∎

We will now in two corollaries study some special cases of physical interest, the first concerning local Maxwellian distribution functions ψ, and the second concerning inverse kth power forces for the interactions.

Corollary 4.3. Assume $B(\theta, w)$, $\hat{\theta}(\hat{R}, w)$, and $R(\mathbf{x}, \mathbf{v} \to \mathbf{v}')$ as in Theorem 4.2. Suppose

$$\psi(\mathbf{x}, \mathbf{v}_*) = X(\mathbf{x}) \cdot \exp(-cm_* v_*^2)$$

and

$$E(\mathbf{v}) = a \cdot \exp(-cmv^2)$$

with continuous function $X(\mathbf{x}) > 0$ and positive constants a, c, m_*, and m, such that the detailed balance relation (4.11) holds. If $(1+v)^{q_0} F_0(\mathbf{x}, \mathbf{v}) \in L^1_+(D \times V)$ for some $q_0 > 2$, and $F_0(\mathbf{x}, \mathbf{v}) \log F_0(\mathbf{x}, \mathbf{v}) \in L^1(D \times V)$, then (for $t > 0$) there exists a solution $f(\mathbf{x}, \mathbf{v}, t) \in L^1_+(D \times V)$ to the linear Boltzmann equation in the integral form (4.1) with infinite-range forces. The solution satisfies (4.13) and (4.14) for $0 \leqslant \lambda < 2$, and also (4.17), if $1 \leqslant \lambda < 2$.

Proof. One finds that

$$\begin{aligned} F_0 \log(F_0/E) &= F_0 \log F_0 - F_0 \log E \\ &= F_0 \log F_0 + (cmv^2 - \log a) F_0 \in L^1(D \times V) \end{aligned}$$

and the corollary follows from Theorem 4.2. ∎

Corollary 4.4. Assume in the case of infinite-range, inverse kth power forces with $k > 3$ that ψ, F_0, and R satisfy (4.7)–(4.12) and (4.16) with $\lambda = (2k-6)/(k-1)$. Then (for $t > 0$) there exists a solution $f(\mathbf{x}, \mathbf{v}, t) \in L^1_+(D \times V)$ to the integral equation (4.1). The solution conserves mass. Higher moments

$$\int_D \int_V (1+v^2)^{q/2} f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v}$$

are globally bounded in time and satisfy (4.17) if $k \geqslant 5$, respectively locally bounded (i.e., for $0 \leqslant t \leqslant T < \infty$, T arbitrary) with (4.14) if $3 < k < 5$, provided they exist initially.

Proof. See (0.10) with $\gamma = \lambda - 1$, and use Theorem 4.2. ∎

Finally we will formulate an H-theorem for our L^1-solutions in the infinite-range case, following a method used by Elmroth[7] for the nonlinear, space-homogeneous equation. We start with the following useful lemma with the H-functional H_E defined in (2.1).

Lemma 4.5. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $L^1_+(D \times V)$ with $f_n \log f_n \in L^1(D \times V)$. If f_n tends to f weakly in $L^1_+(D \times V)$, and $f_n \log E$ converges weakly to $f \log E$, then

$$H_E(f) \leqslant \liminf_{n \to \infty} H_E(f_n)$$

Proof. (Cf. Theorem 13 in ref. 8 or Theorem 6 in ref. 7.) Since $f_n \log(f_n/E) = f_n \log f_n - f_n \log E$, then the lemma is an immediate consequence of the following proposition with $\varphi(f) = f \log f$.

Proposition A.[6] Let $\varphi: \mathbb{R} \to \mathbb{R}$ be a measurable function. Define, for $\Omega \subset \mathbb{R}^3 \times \mathbb{R}^3$, a functional on L^p, $1 \leqslant p < \infty$, by

$$H^{\varphi}(f, \Omega) = \iint_{\Omega} \varphi(f(\mathbf{x}, \mathbf{v}))\, d\mathbf{x}\, d\mathbf{v}$$

Then, for each measurable $\Omega \subset \mathbb{R}^3 \times \mathbb{R}^3$, H^{φ} is lower semicontinuous with respect to weak L^p-convergence if and only if φ is a convex function.

Then we get the following version of an H-theorem in the infinite-range case.

Theorem 4.6. Suppose $f = f(\mathbf{x}, \mathbf{v}, t)$ is a solution to the linear Boltzmann equation in integral form (4.1) given by Theorem 4.2. Then the relative H-functional $H_E(f)(t)$ exists and satisfies (for $t > 0$)

$$H_E(f)(t) \leqslant H_E(F_0)$$

Proof. We start from the cutoff case with cutoff radius $\hat{R} = n$. By Theorem 2.1

$$H_E(f^n)(t) \leqslant H_E(F_0) + \int_0^t [N_E(f^n)(\tau) + N_E^b(f^n)(\tau)]\, d\tau \leqslant H_E(F_0)$$

Then by Theorem 4.2 (and Lemma 4.1) there exists a subsequence $\{f^{n_j}\}_{j=1}^{\infty}$, such that $f = w - \lim_{j \to \infty} f^{n_j}$ exists, and, by Lemma 4.5,

$$H_E(f)(t) \leqslant \liminf_{j \to \infty} [H_E(f^{n_j})(t)] \leqslant H_E(F_0) \quad \blacksquare$$

Remark. Actually, we get a slightly better result,

$$H_E(f)(t) \leqslant H_E(F_0) + \liminf_{j \to \infty} \left\{ \int_0^t [N_E(f^{n_j})(\tau) + N_E^b(f^{n_j})(\tau)]\, d\tau \right\}$$

with $N_E \leqslant 0$ and $N_E^b \leqslant 0$ defined in (2.5), (2.6).

5. THE CASE OF AN UNBOUNDED SPACE DOMAIN

In this section we will discuss generalizations of the results from Sections 1–4 to an unbounded, (strictly) convex domain D.

The existence theorem, Theorem 1.3 in Section 1, holds also for unbounded D, using the same construction of solution by iterates (1.14), where $t_b = t_b(\mathbf{x}, \mathbf{v}) = \infty$, if the right-hand side of (1.7) is not defined. To get an analogy of the detailed balance relation (1.27), (1.29), we assume that there exists a function $E = E_t(\mathbf{x}, \mathbf{v}) \equiv E_0(\mathbf{x} - t\mathbf{v}, \mathbf{v}) > 0$, such that

$$K(\mathbf{x} + t\mathbf{v}, \mathbf{v} \to \mathbf{v}')\, E_0(\mathbf{x}, \mathbf{v}) = K(\mathbf{x} + t\mathbf{v}, \mathbf{v}' \to \mathbf{v})\, E_0(\mathbf{x}, \mathbf{v}') \tag{5.1}$$

or

$$\psi(\mathbf{x} + t\mathbf{v}, \mathbf{v}_*)\, E_0(\mathbf{x}, \mathbf{v}) = \psi(\mathbf{x} + t\mathbf{v}, \mathbf{v}'_*)\, E_0(\mathbf{x}, \mathbf{v}') \tag{5.2}$$

Then

$$(QE_t)(\mathbf{x} + t\mathbf{v}, \mathbf{v}, t) \equiv 0$$

and

$$\frac{d}{dt}\,(E_t(\mathbf{x} + t\mathbf{v}, \mathbf{v})) \equiv \frac{d}{dt}\,(E_0(\mathbf{x}, \mathbf{v})) = 0$$

so $E = E_t(\mathbf{x}, \mathbf{v})$ satisfies Eq. (0.1).

For the boundary we assume that the same detailed balance relation (1.30) holds as in Sections 1–4, but now with

$$E^b(\mathbf{x}, \mathbf{v}) = \lim_{s \to 0\pm} E_0(\mathbf{x} - s\mathbf{v}, \mathbf{v}), \qquad \mathbf{x} \in \Gamma_\pm(\mathbf{v}) \tag{5.3}$$

Then $E = E_t(\mathbf{x}, \mathbf{v})$ is a (collision invariant) solution to the linear Boltzmann equation in the strong form (0.1) with (0.2) and (0.3), and also to the equation in the mild form (1.9) and in the exponential form (1.10), if $F_0(\mathbf{x}, \mathbf{v}) = E_0(\mathbf{x}, \mathbf{v})$. Here we assume that

$$E_0(\mathbf{x}, \mathbf{v}) \in L^1_+(D \times V)$$

A physically interesting case is given by locally Maxwellian functions $E_t(\mathbf{x}, \mathbf{v})$, where

$$E_0(\mathbf{x}, \mathbf{v}) = a \exp(-cmv^2 - bx^2)$$

with constants $a, b, c > 0$, if

$$\psi(\mathbf{x}, \mathbf{v}_*) = X(\mathbf{x}) \exp(-cm_* v_*^2)$$

The statements in Theorem 1.4 about mass conservation and uniqueness hold if

$$K(\mathbf{x} + t\mathbf{v}, \mathbf{v} \to \mathbf{v}')\, E_0(\mathbf{x}, \mathbf{v}) \in L^1_+(D \times V \times V)$$

and

$$|\mathbf{nv}|\ E_0(\mathbf{x},\mathbf{v}) \in L^1_+(\Gamma \times V)$$

For the proof, use that the iterates satisfy

$$f_n^p(\mathbf{x}+t\mathbf{v},\mathbf{v},t) \leqslant p \cdot E_0(\mathbf{x},\mathbf{v}), \qquad n \in \mathbb{N}$$

if

$$F_0^p(\mathbf{x},\mathbf{v}) = \min(F_0(\mathbf{x},\mathbf{v}),\ pE_0(\mathbf{x},\mathbf{v})), \qquad p = 1, 2, 3, ...$$

The H-theorem, Theorem 2.1 in Section 2, holds also for an unbounded domain D, with the function $E_0(\mathbf{x},\mathbf{v})$ in (2.5) and (2.6) instead of $E(\mathbf{v})$, and $E_0(\mathbf{x},\mathbf{v}')$ instead of $E(\mathbf{v}')$, after a change of variables $\mathbf{x} \mapsto \mathbf{x}+t\mathbf{v}$. For the proof use the functions [cf. (2.11a)]

$$f_n^{p,j}(\mathbf{x},\mathbf{v},t) = f_n^p(\mathbf{x},\mathbf{v},t) + \frac{1}{j} E_t(\mathbf{x},\mathbf{v}), \qquad n \in \mathbb{N}, \quad p, j = 1, 2, 3, ...$$

The local boundedness proposition about higher moments, Proposition 3.1 in Section 3, has an analogy for an unbounded domain D,

$$\int_D \int_V (1+x^2+v^2)^{q/2} f(\mathbf{x},\mathbf{v},t)\ d\mathbf{x}\, d\mathbf{v}$$
$$\leqslant e^{At} \int_D \int_V (1+x^2+v^2)^{q/2}\, F_0(\mathbf{x},\mathbf{v})\, d\mathbf{x}\, d\mathbf{v}$$

for $0 < q \leqslant q_0$ and some constant A, if F_0 satisfies

$$(1+x^2+v^2)^{q_0/2}\, F_0(\mathbf{x},\mathbf{v}) \in L^1_+(D \times V) \tag{5.4}$$

and if the other assumptions in Proposition 3.1 are satisfied. For a proof, start with (3.6) and multiply by $(1+v^2+|\mathbf{x}+t\mathbf{v}|^2)^{q/2}$ with a suitable cutoff [cf. $h_{q,r}$ in (3.1)].

In the noncutoff case, including infinite-range forces (cf. Section 4), there exist L^1-solutions to the integral equation (4.1) even in the case of unbounded space domain D (cf. Theorem 4.2). First we get an analogy to Lemma 4.1, by changing $(1+v)$ to $(1+x^2+v^2)^{1/2}$ in (4.1)–(4.4) and using a function $E(\mathbf{x},\mathbf{v})$ instead of $E(\mathbf{v})$ in (4.3). Then we can prove a generalization of Theorem 4.2 to unbounded domains D, including existence, mass conservation, and local boundedness of higher moments for both soft and hard collision potentials, $0 < \lambda < 2$.

Theorem 5.1. Let the assumptions on B, $\hat{\theta}$, and ψ in Theorem 4.2 be satisfied [cf. (4.5)–(4.8)], together with (5.4) for some $q_0 > \lambda$, $q_0 \geqslant 1$. Let

$$F_0(\mathbf{x}, \mathbf{v}) \log[F_0(\mathbf{x}, \mathbf{v})/E_0(\mathbf{x}, \mathbf{v})] \in L^1(D \times V)$$

where $E_0(\mathbf{x}, \mathbf{v}) > 0$ with

$$\{\log[E_0(\mathbf{x}, \mathbf{v})]\}/(1 + x^2 + v^2)^{\bar{q}/2} \in L^\infty(D \times V)$$

for some $\bar{q} < q_0$, and $E_0(\mathbf{x}, \mathbf{v})$ satisfies the detailed balance relations (5.1) and (1.30) with (5.3), and let (4.12) hold.

Then there exists (for all $t > 0$) a nonnegative solution $f = f(\mathbf{x}, \mathbf{v}, t) \in L^1_+(D \times V)$ to the linear, space-inhomogeneous Boltzmann equation in integral form (4.1) with unbounded, (strictly) convex body D. The solution satisfies

$$\int_D \int_V f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v} = \int_D \int_V F_0(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v}$$

and

$$\int_D \int_V (1 + x^2 + v^2)^{q/2} f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v}$$
$$\leqslant e^{At} \int_D \int_V (1 + x^2 + v^2)^{q/2} F_0(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v}$$

for $0 < q \leqslant q_0$ (with some constant A), and for $0 \leqslant t \leqslant T < \infty$, where T is arbitrary.

Proof. Use the generalization of Lemma 4.1 mentioned above, and essentially the same technique as in the proof of Theorem 4.2. ∎

Finally, we can get natural analogies to the physically interesting cases, Corollary 4.3 and 4.4, and also get an H-theorem for unbounded domains in the noncutoff case (cf. Theorem 4.6).

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